SCOTT, FORESMAN EXPERIMENTAL MATHEMATICS PROJECT

TEACHER'S GUIDE
FOR
BOOKLET

The System of Natural Numbers

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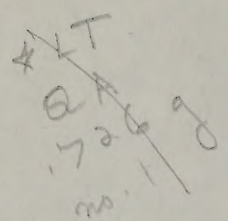

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PART ONE: INTRODUCTION

BOOKLET 7 is part of the Seeing Through Mathematics program for seventh and eighth grades. This program is a carefully thought out departure from traditional seventh- and eighth-grade mathematics. It is a program in mathematics rather than a review of the arithmetic of the first six grades. The mathematical concepts developed in the course are of a fundamental and powerful nature and are presented in such a fashion that they are well within the range of comprehension of seventh- and eighth-grade students. This is accomplished by a continual recognition of, and rigorous adherence to, the principles of the psychology of learning.

The reports of various committees of mathematicians and educators have indicated unanimously that a drastic revision is needed in mathematics at the elementary and high school levels. These reports have emphasized that (1) mathematical ideas are not fully understood by students and consequently are not retained for any appreciable period of time; (2) mathematics is not enjoyed by some students, hence these students tend to avoid courses in mathematics; and (3) the spirit of school mathematics is not in keeping with the dynamic, creative mathematics of today's world.

For these reasons it is clear that a departure from the traditional mathematics program is needed. After careful study and thoughtful research, Scott, Foresman and Company is preparing a program in mathematics designed so that (1) students will understand mathematical concepts and will retain them; (2) students will enjoy the intellectual challenge of mathematics and will experience satisfying success in the field; and (3) the creative, growing spirit of today's mathematics will be brought into the school.

As you study BOOKLET 7, you will notice that traditional ideas are presented along with many new ideas. The traditional content, however, is studied from a new and different point of view. This new viewpoint, based upon a few fundamental ideas, provides a broad general framework into which the traditional concepts fit in a unified fashion. Formerly isolated and unrelated, the traditional ideas now have a general frame of reference. A general

perspective of the field of mathematics is obtained. This perspective enables the student to fit the parts together intelligently. In this program, a groundwork is laid that will enable the student to see advanced mathematics from this same perspective.

The major units included in the seventh-and eighth-grade programs are listed below:

Unit 1: Algebraic and geometric conditions in one variable (including problem solving)

Unit 2: Algebraic and geometric compound conditions in one variable (including problem solving)

Unit 3: Simple and compound conditions in two variables (including problem solving)

Unit 4: Proportional relations (including work with ratios, rate pairs, and per cents)

Unit 5: Numeration systems for natural numbers

Unit 6: The system of natural numbers

Unit 7: The system of non-negative rational numbers (fractions), (including conditions and problem solving involving non-negative rational numbers)

Unit 8: Lines, angles, plane figures, and area

Unit 9: Simple and compound conditions in one, two, and three variables

Unit 10: A review and extension of the work with natural numbers (including number theory, permutations and combinations, and natural number exponents)

Unit 11: A review and extension of the work on the non-negative rational numbers (including numeration systems in other bases for fractions, and some work with probability)

Unit 12: The system of rational numbers (including the negative rationals along with zero and the positive rationals)

Unit 13: Irrational numbers (a brief unit on irrational numbers to include square root)

Unit 14: Mathematical systems (a brief unit to emphasize that different mathematical systems have different structure)

Unit 15: Formulas (a further unit on conditions to include formulas and inverse variation)

Unit 16: Descriptive statistics and graphing

Unit 17: Geometry

Unit 18: Algebra of polynomial expressions

Unit 19: Equivalent conditions in one variable

The first several units of work involve the natural numbers only. The reason for this is that we are developing some very powerful and some very important mathematical concepts. We want to do this in as simple a framework as possible. Thus, we develop the notion of a condition, the notion of a variable, the notion of a set, the notion of an ordered pair, the notion of a relation, and a notion of a number system with just the natural numbers. Later on, as we extend the system of natural numbers to the system of non-negative rational numbers, and finally, to the system of real numbers, we will extend each of these ideas to include numbers other than the natural numbers.

The nineteen units that are described above form an integrated program in mathematics for Grades 7 and 8 based on fundamental mathematical ideas. It is somewhat misleading to give a list of units, because the reader may conclude that each of these units is an isolated topic in itself. This is not the case. Each idea developed in each of the units is continued and used in the development of other units. Certain units emphasize, for example, work on geometry, but ideas from geometry are developed throughout the program. The work on geometry involves a set-theoretic approach. The emphasis is on geometric ideas and intuition. There is considerable work on non-metric geometry. This involves set terminology in the framework of some of the concepts of geometry that do not involve measurement. Lines, planes, and space are regarded as sets of points. The properties of intersections of these point sets are included. Separation properties of space by planes, of planes by lines, and of lines by points are considered.

As we have mentioned, these nineteen units form the basis of the content of the Seeing Through Mathematics program for Grades 7 and 8. The first book in this program is Seeing Through Mathematics 1 (STM 1), and the second book is Seeing Through Mathematics 2 (STM 2). STM 1 is designed for Grade 7 and STM 2 for Grade 8. The first eight units above form the content of STM 1, and the remaining units the content of STM 2.

Observe the continuity of the program. Each unit in STM 1 is reviewed and extended in STM 2. The experimental booklets that have been prepared include material that covers content from both STM 1 and STM 2. It is for this reason that the booklets have been tried out in both Grade 7 and Grade 8. The reactions of our project teachers have been very helpful in deciding which content to include in STM 1 and which content to include in STM 2.

BOOKLET 7 is on the system of natural numbers. Most of the content included in BOOKLET 7 is in STM 1. Some of the content of BOOKLET 7, however, appears in the work on natural numbers in STM 2.

BOOKLET 7 is an excellent booklet to use with eighth-grade students who have not studied STM 1. It will help to clarify their ideas about numbers and to stimulate interest in learning more about numbers. It should be followed in its use by BOOKLETS 8, 9, 10, and 11. BOOKLETS 8, 9, 10, and 11 cover content from both STM 1 and STM 2.

Practice in computation is included in these booklets only when it is necessary preparation for lessons in which much computation is employed. However,it is extremely important for all students to be able to compute with accuracy and skill. Consequently,STM 1 and STM 2 will include the work on computation that is included at the seventh-grade level and the eighth-grade level in most standard textbooks.

Maintenance material is included in these booklets only when it is a necessary preparation for the lessons that follow. STM 1 and STM 2 will include a great deal of maintenance material. The basic skills of arithmetic will be maintained, as well as the new material after it is introduced.

The amount of practice material in the problem-solving lessons of these experimental booklets will also be increased for the problem-solving

lessons in the hard-cover books. Because of time and space limitations, we have included only a minimum amount of practice and maintenance in the booklets.

Business applications of mathematics will be a vital part of the STM program. Problems concerning discount, interest, commission, etc., are already included in the work with ratio and with per cent in BOOKLET 4. The seventh-grade book will include all kinds of problems from science and industry and business as applications of the material that is studied. The difference between our program and other programs concerning these topics is that we will not build a program around social applications. Our program is being structured around basic mathematical ideas, and these ideas will be applied to problems from different areas of life.

The inductive approach of presentation is used in the booklets and will be used in the STM textbooks. Therefore, it is necessary to ask questions and to include somewhat more reading material than in the traditional kinds of materials that teachers are used to in mathematics. However, the teacher should feel free to eliminate some of the discussion material as he presents the material in class. When two sets of developmental exercises relating to the development of one concept are presented, it may be very appropriate in class to use only one set of exercises. In other words, the teacher should select the developmental material he thinks is appropriate for his own individual class. Likewise, the teacher should feel free to use the "On Your Own" exercises in any way he chooses. Often the teacher may not wish to assign all of these exercises to his individual class.

In traditional material, the method of presentation is very descriptive and is a method of telling the student something and then asking him to work problems relating to the topic being studied. In the booklets and in the textbook, we will continue to use a method of guided discovery. This method will stimulate the student into thinking for himself and will also give him an opportunity to experience the excitement and stimulation of making generalizations for himself.

A teacher who wishes to use BOOKLET 7 need not be afraid that he has an insufficient mathematical background. A serious study of this Teaching

Guide along with the lessons is sufficient for a basic understanding of the concepts and problems developed in the booklet. This Teaching Guide includes four parts in addition to the Introduction. Part Two, "Background Information," presents a brief discussion of the ideas that are developed in BOOKLET 7. It also relates the material included in BOOKLET 7 to the material covered in the previous booklets and to the material that will follow the work on the system of natural numbers. Part Three, "Charting the Course," summarizes the content of each lesson and suggests a time schedule for teaching the lessons. Part Four, "Lesson Briefs," contains helps for teaching each lesson, as well as answers to some of the developmental exercises and all the "On Your Own" exercises. Part Four also includes a "lesson by lesson" discussion of the mathematical ideas in each lesson. It was added at the suggestion of many of the participating teachers. Part Five, "Appendix," includes an index of definitions and a glossary and index of symbols. This Teaching Guide should be used in conjunction with BOOKLET 7. Refer to the appropriate lesson in BOOKLET 7 as you read the Guide.

PART TWO: BACKGROUND INFORMATION

Before you begin work on the system of natural numbers in BOOKLET 7, you may want to consider the work on the system of natural numbers relative to the entire sequence of topics. Part One of this Teaching Guide contains a topical outline of the seventh- and eighth-grade programs. You will note that this outline has been changed from the outline that has been presented in previous Teaching Guides.

The first eight units of the topical outline will form the content of Seeing Through Mathematics, Book 1; the remaining units will form the content of Seeing Through Mathematics, Book 2. The last unit in the seventh-grade program will include work on lines, angles, plane figures, and area. The first unit in the eighth-grade program will be on conditions in one, two, and three variables. This unit will review some of the work of the seventh-grade program, as well as extend

the work on conditions to three variables. The conditions in three variables will include many familiar formulas from business, science, and geometry. Thus, the work on geometry that is included as the last unit in the seventh-grade program will be extended in the first unit in the eighth-grade program.

You will observe from the topical outline in Part One that our basic belief is that mathematics is one subject, best learned as one subject whose parts reinforce each other and advance together. This belief is carried out through a unified mathematics program that is modern in content, in pedagogy, and in spirit. The program is so organized as to promote the view of mathematics as a single subject, not a collection of isolated techniques. It is for this reason that it is of major importance that the teacher have a view of the entire program, and be very sure that he sees just where any particular unit fits into the structure of the entire program. You will note, as you think about the program as a whole, that judicious and careful use has been made of such unifying ideas as set, variable, relation, and number system.

You will observe from the topical outline that the work on the system of natural numbers immediately follows the unit on numeration systems. It follows a substantial part of the book devoted to sets, conditions, and their applications to problem solving. The work done on problem solving prepares the student for the kind of thinking that he will have to do in the work on numeration systems. It must be emphasized that, in our view, problem solving is the unique form of mathematical activity by which a student acquires the values and powers of mathematics. It is for this reason that problem-solving activity is emphasized in our program. It is through problem-solving activity that a student has the chance to become acquainted with the procedures of mathematical experimentation and discovery that are seen in actual mathematical research. The work on numeration systems involves on the part of the student the kind of thinking and discovery that is necessary in mathematical research.

The work on numeration systems was included before the work on the system of natural numbers because the student is more familiar with numeration systems than he is with the abstract concept of a number system. He has had considerable work with numeration systems in elementary arithmetic. However,

it is very probable that numeration systems, or systems for writing names for numbers, were called number systems in elementary arithmetic. Since, in our program, we are constantly aware of the name-referent distinction, we have been very careful in preserving this distinction when it comes to numeration systems and number systems. This is merely an extension of the name-referent distinction as applied to the numeral-number distinction.

In the past, properties of numerals and properties of numbers, and properties of numeration systems and properties of number systems were completely interchanged and confused. Many properties that rightfully belonged to numbers were attributed to numerals, and vice versa. Thus, the student could never be completely certain as to which properties applied to numbers and which applied to numerals.

The basic objectives of BOOKLET 7 are to develop the properties of natural numbers and to define a number system. If the student is able to realize these objectives, he will have a better understanding of the numbers he uses every day.

In Lessons 1 through 3, the concepts of mapping and one-to-one correspondence are introduced, and some of the properties of natural numbers are developed. In Lessons 4 through 8, the operations of addition and multiplication are defined. The commutative property and the associative property for each of the operations are developed as is the distributive property. The remaining lessons are devoted to the requirements necessary for a number system and to the development of other special properties of the natural numbers.

It is important that the students thoroughly understand the material presented in the first three lessons of BOOKLET 7. The concepts presented in these lessons form a foundation upon which other concepts are developed.

A set is made up of objects that are its members. These objects may be alike or they may be different. Consider the set of natural numbers or the set of letters in the English alphabet. The objects in each of these sets are alike. However, the Eskimo huts in Alaska and the residences in Hollywood, California, can constitute a single set should anyone wish to work with such a

set. It is possible that a set may not have any members. This set is called the empty set. Examples of the empty set are the set of automobiles made in the year 800, and the set of college students who are under 6 years of age.

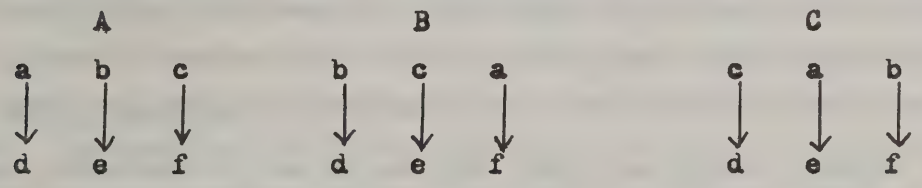
In BOOKLET 7, a natural number is defined as a property of certain sets. These sets must be related to each other by setting up a one-to-one correspondence between them. Preliminary to one-to-one correspondence, the concept of mapping is introduced. Mapping is considered first because this concept leads to the idea of one-to-one correspondence, and because various types of mappings can be used in explaining many mathematical ideas.

Matching the members of a set with members of another set according to some specified plan is called <u>mapping</u>. Consider the sets M and N.

M = {a, b, c}.

N = {d, e, f}.

A one-to-one mapping of set M <u>onto</u> set N is a matching of each member of M with a member of N in such a way that all members of both sets are used. Study the one-to-one mappings shown under A, B, and C. Three different mappings are shown. Remember that the order in which the members are listed does not change the set. The arrow is used to show which member of M is mapped onto which member of N.



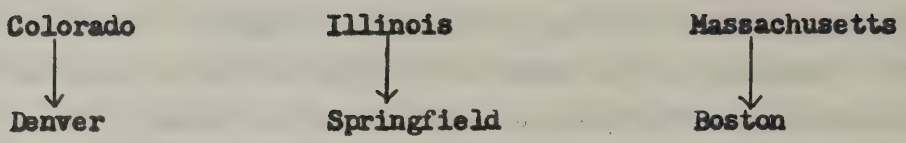
Mappings can also be many-to-one mappings. Consider the pairs of natural numbers whose sum is 5. The sum of the components of the ordered pairs (0,5),(1,4), (2,3), (3,2), (4,1),and (5,0) are each equal to 5, and, therefore, all of these ordered pairs may be mapped by addition onto the natural number 5.

Two one-to-one mappings are necessary to make a one-to-one correspondence. Consider sets C and D.

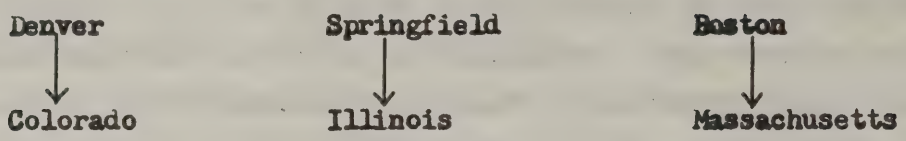
C = {Colorado, Illinois, Massachusetts}.

D = {Denver, Springfield, Boston}.

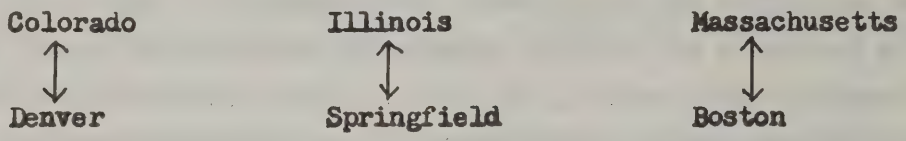
First, each state is mapped onto its capital city.



Next, each capital city is mapped onto its state.



Notice that in both mappings the same members of each set are linked together. In a one-to-one correspondence, the "mate" of a particular member of either set is uniquely determined by either of the mappings from which it is structured. The two mappings above form a one-to-one correspondence. A one-to-one correspondence, made up of two mappings combined, is usually described in writing by means of a double-headed arrow.



The two sets, C and D, are in one-to-one correspondence because there is a one-to-one mapping of the members of set C onto the members of set D, and there is a one-to-one mapping of the members of set D onto the members of set C. Any two sets that can be put in one-to-one correspondence are <u>equivalent sets</u>.

Since {oriole, robin} and {elm, oak} can be put in one-to-one correspondence, they are equivalent sets. Moreover, any set that can be put in one-to-one correspondence with {elm, oak} is equivalent to {elm, oak}. All finite sets that are equivalent to each other are said to have the same natural number. Therefore, a natural number is a common property of equivalent sets. The natural number of a set that is not infinite tells how many members are in the set.

Each natural number is associated with a particular set and all sets equivalent to the particular set. The particular set to which each natural number is associated is called the <u>standard set</u> of that number.

The empty set, { }, is the standard set that has no members. The natural number 0 is associated with the empty set and tells how many members are in this standard set. In mathematical language, this means that $0 = \underline{n}\{\ \}$. This sentence is read: "The natural number 0 equals the number of members in the empty set." Any set that is equivalent to the empty set is associated with the natural number 0. The set of automobiles made in the year 800 is equivalent to the empty set. Therefore, the number of automobiles made in the year 800 is 0.

Now that the natural number 0 has been defined, 0 can be used as a member of a standard set. The standard set {0} has one member, namely 0. The natural number 1 is defined as the number of members in {0} and is associated with the standard set {0}. The number associated with any set that is equivalent to {0} is 1.

Since the natural numbers 0 and 1 have been defined, they can now be used as members of a standard set, namely {0, 1}. The natural number 2 is defined as the number of members in the standard set {0, 1}. The number associated with any set equivalent to {0, 1} is 2.

This process can be continued so that each natural number can be defined using a standard set. The standard set of a natural number is made up only of those numbers that have already been developed. For example, $5 = \underline{n}\{0, 1, 2, 3, 4\}$; $7 = \underline{n}\{0, 1, 2, 3, 4, 5, 6\}$; $12 = \underline{n}\{0, 1, 2, \cdots, 11\}$, and so on. Furthermore, each set that exists, and is not a standard set, is equivalent to a standard set and has the same natural number as the standard set. The set {a, e, i, o, u} is equivalent to the standard set {0, 1, 2, 3, 4}. Therefore, the set {a,e, i, o, u} is associated with the natural number 5. The set of natural numbers is defined as follows:

$$N = \{0, 1, 2, \cdots\}.$$

The natural numbers have the property of order. This means that, given any two natural numbers, either one is greater than the other or they are equal; however, both of these cannot be true at the same time. The ordering of natural numbers can be accomplished by means of sets. 4 is greater than 3

because the standard set of 3 is a proper subset of the standard set of 4. A proper subset is any subset that does not contain all of the members of a specified set. The statements $4 > 3$ and $3 < 4$ imply the same idea.

The natural numbers also have the successor property and the property of betweenness. The successor property of natural numbers states that, given any natural number, there is a natural number that immediately follows it. For example, 17 is the successor of 16; 40 is the successor of 39; 1732 is the successor of 1731; and so on. Each natural number except 0 is a successor of another natural number.

Since the definition "between" is understood, the property of betweenness means that, given any two different natural numbers, there is a finite set of natural numbers "between" the given natural numbers. Consider the compound condition $\underline{a} > \underline{x} \wedge \underline{x} > \underline{b}$. If $\underline{a}$ and $\underline{b}$ are replaced by two different members of N, and the replacement for $\underline{a}$ is greater than the replacement for $\underline{b}$, then after replacement of $\underline{a}$ and $\underline{b}$ the condition $\underline{a} > \underline{x} \wedge \underline{x} > \underline{b}$ will have a finite solution set. For example, if $\underline{a}$ is replaced by 8 and $\underline{b}$ is replaced by 2, we have the condition $8 > \underline{x} \wedge \underline{x} > 2$. The solution set of this condition will be $\{3, 4, 5, 6, 7\}$. If $\underline{a}$ is replaced by 5 and $\underline{b}$ is replaced by 4, the solution set of the condition $5 > \underline{x} \wedge \underline{x} > 4$ is the empty set.

At this point, it may be well to consider the relations that are important to the development of the natural numbers. A relation is defined as a set of ordered pairs. It must be emphasized that this precise definition of relation differs from the everyday meaning of relation. Some of the relations which are important in mathematics are those of "equals," "is greater than," "is less than," and "is not equal to." Relations have properties. The properties of relations that are most commonly used by mathematicians are the symmetric property, the reflexive property, and the transitive property.

First consider the symmetric property. The relation of "equals" has the symmetric property. This means that, if $\underline{a} = \underline{b}$ yields a true statement after replacement of the variables, then $\underline{b} = \underline{a}$ will yield a true statement with the same replacements. For example, both $7 \times 5 = 35$ and $35 = 7 \times 5$ are true statements; both $\underline{x} + 5 = 24$ and $24 = \underline{x} + 5$ will yield true statements when $\underline{x}$ is

replaced by 19. The relation "is not equal to" is also symmetric. For example, $6 \neq 7$ and $7 \neq 6$ are true statements. The relations of "is greater than" and "is less than" are not symmetric. $9 > 5$ is a true statement, but $5 > 9$ is a false statement. Likewise, $4 < 8$ is a true statement, but $8 < 4$ is a false statement.

Students usually recognize the symmetric property without having it brought to their attention. To express the same condition, students may write either the sentence $\underline{x} - 5 = 2$ or the sentence $2 = \underline{x} - 5$. The students should not be instructed to use one sentence in preference to the other as both sentences express the same condition. However, in writing certain sentences, custom dictates a conformity that should be followed. For example, in naming sets, write $A = \{0, 1, 2\}$ and not $\{0, 1, 2\} = A$. Write $4 = \underline{n}\{0, 1, 2, 3\}$ and not $\underline{n}\{0, 1, 2, 3\} = 4$. Do not expect the students to use the same forms that you would use.

Next consider the reflexive property. To explain the reflexive property, consider a relation that is represented by the symbol '*'. In mathematical language, the relation '*' is reflexive if $\underline{a} * \underline{a}$ yields a true statement for all replacements of the variable. The relation of "equals" exhibits the reflexive property. All of the following are true statements: $3 = 3$; $2 + 4 = 2 + 4$; $5 \times 8 = 5 \times 8$, and so on. The false statements $3 \neq 3$; $3 > 3$; and $3 < 3$ show why the relations of "is not equal to," "is greater than," and "is less than" are not reflexive.

Finally consider the transitive property. The relation of "equals" is transitive. This means that, if $\underline{a} = \underline{b}$ and $\underline{b} = \underline{c}$ yield true statements after replacement of the variables, then $\underline{a} = \underline{c}$ will yield a true statement with the same replacements. For example, $4 + 5 = 11 - 2$ and $11 - 2 = 9$ are true statements; therefore, $4 + 5 = 9$ is a true statement. The relations of "is greater than" and "is less than" are also transitive. Consider the following examples. $12 > 6$ and $6 > 4$ are true statements; therefore, $12 > 4$ is also a true statement. $3 < 19$ and $19 < 27$ are true statements; therefore, $3 < 27$ is also a true statement. The relation of "is not equal to" is not transitive. The following example should show why the transitive property does not hold: $8 - 2 \neq 4$ and $4 \neq 6$ are true statements, but $8 - 2 \neq 6$ is a false statement.

Thus far, it has been shown that the natural numbers have the property of order, the successor property,and the property of betweenness. It has also been shown how relations have special properties. Next the operations of addition and multiplication will be discussed.

Now that we have a set of objects, namely the natural numbers, we may define binary operations on N. A binary operation is a rule that assigns members of N×N to members of N. The word "binary" comes from the fact that each member of N×N has two components. The operations on N emerge from ways in which sets may be related to each other, and also from ways that sets themselves may be operated on.

Two binary operations on sets are union and intersection. The union of two sets is the set that contains all members that belong to either set. The symbol for union is '$\cup$'. Consider sets A and B.

A = {robin, wren, bluejay}.

B = {oriole, peacock, robin}.

A $\cup$ B = {robin, wren, bluejay, oriole, peacock}.

The intersection of two sets is the set that contains only those members that belong to both sets. The symbol for intersection is '$\cap$'. Again consider sets A and B tabulated above.

A $\cap$ B = {robin}.

The two sets under consideration may or may not have members in common. If two sets have no members in common, they are called disjoint sets. Now consider the union and intersection of sets A and C.

A = {robin, wren, bluejay}.

C = {oriole, peacock}.

A $\cup$ C = {robin, wren, bluejay, oriole, peacock}.

A $\cap$ C = { }.

The intersection of two disjoint sets is always the empty set.

Euler-Venn diagrams are often used by mathematicians to represent relationships between sets. The diagrams are named after two mathematicians, Euler and Venn, who worked extensively on the theory of sets. The Euler-Venn

diagrams may be used to represent the union and intersection of two sets. In each diagram, the rectangular-shaped area represents the universe.

Consider sets F and G.

The universe is the English alphabet.

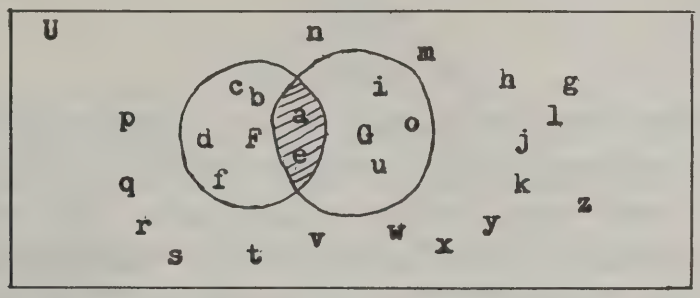
$F = \{a, b, c, d, e, f\}$.

$G = \{a, e, i, o, u\}$.

$F \cup G = \{a, b, c, d, e, f, i, o, u\}$.

$F \cap G = \{a, e\}$.

Note that in $F \cup G$, the common members of the two sets, a and e, are not repeated. In the Euler-Venn Diagram 1 below, the entire area surrounded by circles represents $F \cup G$; the shaded area represents $F \cap G$.



Euler-Venn Diagram 1

Now consider the same set F as used above and set H.

The universe is the English alphabet.

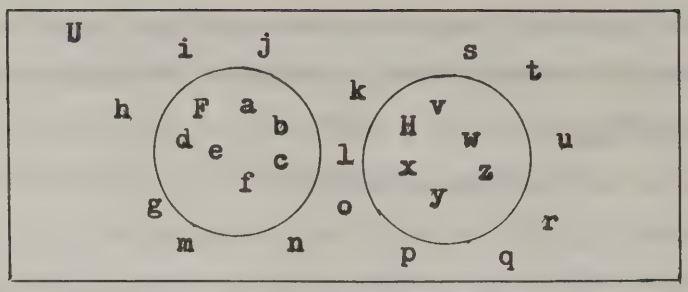
$F = \{a, b, c, d, e, f\}$.

$H = \{v, w, x, y, z\}$.

$F \cup H = \{a, b, c, d, e, f, v, w, x, y, z\}$.

$F \cap H = \{\ \}$.

The Euler-Venn Diagram 2 shows the new union. Notice that the intersection of sets F and H is the empty set. Remember that two sets that have no members in common are disjoint sets.



Euler-Venn Diagram 2

The two examples just given may be used to define a binary operation on N. Using the definition of natural numbers already established, the following express true statements:

$6 = \underline{n}(F)$. $\qquad$ $6 = \underline{n}(F)$.

$5 = \underline{n}(G)$. $\qquad$ $5 = \underline{n}(H)$.

$9 = \underline{n}(F \cup G)$. $\qquad$ $11 = \underline{n}(F \cup H)$.

In the case of the disjoint sets F and H, $\underline{n}(F \cup H)$ is the sum of $\underline{n}(F)$ and $\underline{n}(H)$. This is true only if the sets are disjoint; note that $\underline{n}(F \cup G)$ is not the sum of $\underline{n}(F)$ and $\underline{n}(G)$ because sets F and G are not disjoint sets. The union of disjoint sets is used to define a binary operation on N. This operation is called addition. $\underline{n}(F) + \underline{n}(H) = \underline{n}(F \cup H)$ is a true statement and is the same statement as $6 + 5 = 11$.

The ordered pair of natural numbers (6,5) is mapped by addition onto the natural number $6 + 5$, or 11. Addition is a mapping of a member of $N \times N$ onto a member of N. True statements, such as $0 + 2 = 2$, $5 + 9 = 14$, $3 + 10 = 13$, $39 + 25 = 64$, and so on, result from this mapping. (0,2) is mapped by addition onto 0+2, or 2; (5,9) is mapped by addition onto 5+9, or 14; (3,10) is mapped by addition onto 3+10, or 13; (39,25) is mapped onto 39+25, or 64, and so on.

Addition is a many-to-one mapping of the members of $N \times N$ onto the members of N. For example, (0,4), (1,3), (2,2), (3,1), and (4,0) are mapped by addition onto 4. Since each and every member of $N \times N$ can be mapped by addition onto a member of N, the natural numbers are closed under addition.

The operation of addition can be defined in ways other than the approach depending on the union of disjoint sets. However, this definition seems to be the one most easily expressed in elementary terms.

The second binary operation on N is defined by means of the Cartesian set of two sets that may or may not be disjoint. Consider sets A and B.

$A = \{\Delta, *, \not\Delta\}$. $3 = \underline{n}(A)$.

$B = \{\Delta, \square\}$. $2 = \underline{n}(B)$.

$A \times B = \{(\Delta,\Delta), (\Delta,\square), (*,\Delta), (*,\square), (\not\Delta,\Delta), (\not\Delta,\square)\}$. $6 = \underline{n}(A \times B)$.

Notice that $\underline{n}(A \times B)$ is the product of $\underline{n}(A)$ and $\underline{n}(B)$. The binary operation defined in this way is the operation of multiplication. $\underline{n}(A) \cdot \underline{n}(B) = \underline{n}(A \times B)$ is a true statement and is the same as $3 \cdot 2 = 6$ in this instance. Multiplication is a many-to-one mapping of the members of N×N onto the members of N. The ordered pair (3,2) is mapped by multiplication onto $3 \cdot 2$, or 6. $3 \cdot 2$, or 6, is the product of 3 and 2. Since each and every member of N×N can be mapped by multiplication onto a member of N, the natural numbers are closed under multiplication.

Two other operations on N may be defined using the operations of addition and multiplication. These operations are called inverse operations. Subtraction is the inverse operation of addition and division is the inverse operation of multiplication.

The ordered pair (13,10) is mapped by subtraction onto the number that satisfies $\underline{x} + 10 = 13$. Thus, (13,10) is mapped by subtraction onto the difference 13-10, or 3, because $3 + 10 = 13$ is a true statement. (6,2) is mapped by subtraction onto the difference 6-2, or 4; (20,15) is mapped by subtraction onto the difference 20-15, or 5. However, subtraction of natural numbers is not closed because an ordered pair whose first component is less than the second component cannot be mapped by subtraction onto a natural number. For example, the ordered pair (5,9) is mapped by subtraction onto the natural number that satisfies $\underline{x} + 9 = 5$. There is no natural number that satisfies this condition; therefore, the difference of 5 and 9 is not a natural number.

The ordered pair (6,3) is mapped by division onto the number that satisfies $6 = \underline{x} \times 3$. Thus, (6,3) is mapped by division onto $6 \div 3$, or 2, because

$6 = 2 \times 3$ is a true statement. The number that an ordered pair is mapped onto by division is called a quotient. The ordered pair (18,3) is mapped by division onto $18 \div 3$, or 6; (24,6) is mapped by division onto $24 \div 6$, or 4; (100,10) is mapped by division onto $100 \div 10$, or 10; and so on. Division of natural numbers is not closed because there are some ordered pairs, such as (4,3), (7,12), (10,7), and (13,2), that cannot be mapped by division onto a natural number.

Multiplication and addition are the binary operations that have been defined for the set of natural numbers. Each of these operations has three properties; these are the property of closure, of commutativity, and of associativity. It has been pointed out that the set of natural numbers is closed under addition. Addition of natural numbers is closed because each member of $N \times N$ can be mapped by addition onto a member of N.

Now consider the commutative property of addition. Addition of natural numbers has the commutative property if $\underline{a} + \underline{b} = \underline{b} + \underline{a}$ yields a true statement for all replacements of $\underline{a}$ and $\underline{b}$ by members of N. The commutative property of addition may be proved by using sets and the definition of a sum. Remember that a sum may be defined by the sentence $\underline{n}(A) + \underline{n}(B) = \underline{n}(A \cup B)$. Let A and B be any two disjoint sets, and let $\underline{a}$ be the number of A, and $\underline{b}$ be the number of B. By the definition of sum, $\underline{a} + \underline{b}$ is the natural number associated with $A \cup B$, and $\underline{b} + \underline{a}$ is the natural number associated with $B \cup A$. $A \cup B$ is the same set as $B \cup A$ because these sets have the same members; thus, $\underline{a} + \underline{b}$ is the same number as $\underline{b} + \underline{a}$. Therefore, $\underline{a} + \underline{b} = \underline{b} + \underline{a}$ yields a true statement for all replacements of the variables by members of N, and addition of natural numbers has the commutative property.

Another property to be considered is the associative property of addition. Addition of natural numbers has the associative property if $(\underline{a} + \underline{b}) + \underline{c} = \underline{a} + (\underline{b} + \underline{c})$ yields a true statement for all replacements of $\underline{a}$, $\underline{b}$, and $\underline{c}$ by members of N. Let sets A, B, and C be any three disjoint sets, and let $\underline{a}$ be the number of A, $\underline{b}$ be the number of B, and $\underline{c}$ be the number of C. $(A \cup B) \cup C$ will have the same members as $A \cup (B \cup C)$, and the two sets can be put in one-to-one correspondence. Then by definition of sum, $(\underline{a} + \underline{b}) + \underline{c}$ is the number of $(A \cup B) \cup C$, and $\underline{a} + (\underline{b} + \underline{c})$ is the number of $A \cup (B \cup C)$. Since $(A \cup B) \cup C$ is the same set

as $A \cup (B \cup C)$, $(\underline{a} + \underline{b}) + \underline{c}$ and $\underline{a} + (\underline{b} + \underline{c})$ are the same number. Therefore, $(\underline{a} + \underline{b}) + \underline{c} = \underline{a} + (\underline{b} + \underline{c})$ yields a true statement for all replacements of the variables by members of N, and the operation of addition of natural numbers has the associative property.

It has been shown that the operation of addition of natural numbers is closed and has the commutative property and the associative property. Now consider these properties - closure, commutativity, and associativity - for the operation of multiplication of natural numbers.

You will recall that multiplication of natural numbers is closed because each member of N×N may be mapped by multiplication onto a member of N.

If multiplication of natural numbers has the commutative property, the condition $\underline{ab} = \underline{ba}$ must yield a true statement for all replacements of $\underline{a}$ and $\underline{b}$ by members of N. Select any two sets A and B. Let $\underline{a}$ be the number associated with set A and $\underline{b}$ be the number associated with set B. $\underline{ab}$ will be the number of the Cartesian set A×B, and $\underline{ba}$ will be the number of the Cartesian set B×A. A×B and B×A can be put in one-to-one correspondence by matching the member (a,b) of A×B with the member (b,a) of B×A. Therefore, the natural number of A×B, $\underline{ab}$, is the same as the natural number of B×A, $\underline{ba}$. Since $\underline{ab} = \underline{ba}$ yields a true statement for all replacements of the variables by members of N, multiplication of natural numbers is commutative.

If multiplication of natural numbers has the associative property, the condition $(\underline{a} \cdot \underline{b}) \cdot \underline{c} = \underline{a} \cdot (\underline{b} \cdot \underline{c})$ must yield a true statement for all replacements of the variables by members of N. Select any three sets A, B, and C whose numbers are $\underline{a}$, $\underline{b}$, and $\underline{c}$ respectively. Then $(\underline{a} \cdot \underline{b}) \cdot \underline{c}$ is the number of (A×B)×C, and $\underline{a} \cdot (\underline{b} \cdot \underline{c})$ is the number of A×(B×C). Now if (A×B)×C and A×(B×C) are equivalent (can be put in one-to-one correspondence), then $(\underline{a} \cdot \underline{b}) \cdot \underline{c}$ is the same number as $\underline{a} \cdot (\underline{b} \cdot \underline{c})$.

The members of A×(B×C) are ordered pairs each of whose second component is an ordered pair. The members of (A×B)×C are ordered pairs each of whose first component is an ordered pair. Brackets may be used to write an ordered pair having a component that is an ordered pair. Thus, [(1,2),3] is an ordered pair

whose first component is the ordered pair (1,2) and whose second component is 3. Likewise, [1,(2,3)] is an ordered pair whose first component is 1 and whose second component is the ordered pair (2,3).

To prove the associative law, consider sets A, B, and C.

$A = \{1, 2\}$.

$B = \{3, 4, 5\}$.

$C = \{6, 7, 8\}$.

$(A \times B) \times C$ indicates that, to tabulate the set, the names of the members of $A \times B$ must be found first.

$$A \times B = \{(1,3), (1,4), (1,5), (2,3), (2,4), (2,5)\}.$$

Next the names of the members of the Cartesian set of $(A \times B)$ and C are found.

$$(A \times B) \times C = \{[(1,3),6], [(1,4),6], [(1,5),6], [(2,3),6], [(2,4),6], [(2,5),6], [(1,3),7], [(1,4),7], [(1,5),7], [(2,3),7], [(2,4),7], [(2,5),7], [(1,3),8], [(1,4),8], [(1,5),8], [(2,3),8], [(2,4),8], [(2,5),8]\}.$$

$(A \times B) \times C$ has 18 members. To find the members of $A \times (B \times C)$ the above two steps must be followed. The Cartesian set of A and $(B \times C)$ is tabulated below:

$$A \times (B \times C) = \{[1,(3,6)], [1,(3,7)], [1,(3,8)], [2,(3,6)], \cdots, [2,(5,8)]\}.$$

It can be seen by complete tabulation that $A \times (B \times C)$ also has 18 members. The members of both $(A \times B) \times C$ and $A \times (B \times C)$ are ordered pairs. By matching each member of $(A \times B) \times C$ with each member of $A \times (B \times C)$, the two Cartesian sets can be put in one-to-one correspondence. Since $(A \times B) \times C$ and $A \times (B \times C)$ are equivalent sets, $(\underline{a} \cdot \underline{b}) \cdot \underline{c}$ and $\underline{a} \cdot (\underline{b} \cdot \underline{c})$ are the same number. Therefore, $(\underline{a} \cdot \underline{b}) \cdot \underline{c} = \underline{a} \cdot (\underline{b} \cdot \underline{c})$ yields a true statement for all replacements of the variables by members of N, and multiplication of natural numbers has the associative property.

The distributive property links together the operations of addition and multiplication. Multiplication of natural numbers distributes over addition of natural numbers if $\underline{a}(\underline{b} + \underline{c}) = \underline{ab} + \underline{ac}$ yields a true statement for all replacements of $\underline{a}$, $\underline{b}$, and $\underline{c}$ by members of N. This is another property that may be proved by the use of sets.

Select three sets A, B, and C whose numbers are $\underline{a}$, $\underline{b}$, and $\underline{c}$ respectively, and require that sets B and C be disjoint. $\underline{a}(\underline{b} + \underline{c})$ is the natural number of

$A \times (B \cup C)$, and $\underline{ab} + \underline{ac}$ is the natural number of $(A \times B) \cup (A \times C)$. If $A \times (B \cup C)$ and $(A \times B) \cup (A \times C)$ are equivalent sets, then $\underline{a}(\underline{b} + \underline{c})$ and $\underline{ab} + \underline{ac}$ are the same number.

The following example may be used to prove the distributive law.

$A = \{x, y\}$.

$B = \{z\}$.

$C = \{r, s, t\}$.

$B \cup C = \{z, r, s, t\}$.

$A \times (B \cup C) = \{(x,z), (x,r), (x,s), (x,t), (y,z), (y,r), (y,s), (y,t)\}$.

$A \times B = \{(x,z), (y,z)\}$.

$A \times C = \{(x,r), (x,s), (x,t), (y,r), (y,s), (y,t)\}$.

$(A \times B) \cup (A \times C) = \{(x,z), (y,z), (x,r), (x,s), (x,t), (y,r), (y,s), (y,t)\}$.

Set $A \times (B \cup C)$ and set $(A \times B) \cup (A \times C)$ can be put in one-to-one correspondence. Since sets $A \times (B \cup C)$ and $(A \times B) \cup (A \times C)$ are equivalent sets, the number of $A \times (B \cup C)$ is the same as the number of $(A \times B) \cup (A \times C)$. Thus, $\underline{a}(\underline{b} + \underline{c})$ and $\underline{ab} + \underline{ac}$ are the same number. Since $\underline{a}(\underline{b} + \underline{c}) = \underline{ab} + \underline{ac}$ yields a true statement for all replacements of the variables by members of N, multiplication does distribute over addition.

In BOOKLET 7, the distributive property is not proved, and the set idea is not used. The validity of the distributive property is demonstrated by showing different methods of grouping the same set of objects.

The set of natural numbers, the operations of addition and multiplication of members of this set, and the properties of these operations compose the system of natural numbers. The two operations are closed, commutative, and associative, and multiplication distributes over addition. Any set of objects along with two operations that have the same properties as the system of natural numbers is a number system, and each member of the set of objects is a number. Thus, when mathematicians speak of a number system, they are referring to a set of objects that satisfy certain requirements. However, there is a slight disagreement among mathematicians as to just what requirements are necessary. This book uses the definition that includes the fewest requirements. The requirements of a number system are:

1 A set of objects is included.

2 Two binary operations are defined on the set of objects.

3 Each operation is closed.(Property of closure)

4 Each operation is commutative. (Commutative property)

5 Each operation is associative. (Associative property)

6 One operation distributes over the other. (Distributive property)

After the notion of a number system has been introduced by using the set of natural numbers, finite number systems are considered. Obviously, these examples of a number system involve sets that are finite. A detailed explanation of these examples are given in the Lesson Brief for Lesson 11.

The first set that the students study in Lesson 11 is a set having just two members, E and O. $E = \{0, 2, 4, \cdots\}$ and $O = \{1, 3, 5, \cdots\}$. To obtain the basic addition and multiplication facts for $\{E, O\}$, students simply decide whether or not the sum or product obtained is an even number or an odd number. If $\{E, O\}$ is a number system relative to the operations of addition and multiplication, the properties of a number system must hold true for the two operations. Therefore, students first decide if each operation is closed and then obtain all possible statements from $\underline{a} + \underline{b} = \underline{b} + \underline{a}$, $\underline{ab} = \underline{ba}$, $(\underline{a} + \underline{b}) + \underline{c} = \underline{a} + (\underline{b} + \underline{c})$, $(\underline{ab})\underline{c} = \underline{a}(\underline{bc})$, $\underline{a}(\underline{b} + \underline{c}) = \underline{ab} + \underline{ac}$. These statements are obtained by replacing the variables in the conditions by members of $\{E, O\}$. Since the operations of addition and multiplication are closed in $\{E, O\}$ and each statement obtained is true, $\{E, O\}$ satisfies all of the requirements of a number system.

In Lesson 11, students also work with finite number systems which we call remainder systems. In each system the set involved is composed of all possible remainders resulting from the division of the members of N by a selected member of N. The selected member of N that serves as a divisor is called the modulus. For example, $\{0, 1, 2, 3, 4\}$ contains all possible remainders obtained by dividing the members of N by the number 5. The number 5 is the modulus.

For each remainder system, the operations of addition and multiplication are defined on the set involved. Then the students decide whether or not each set is a number system relative to the two operations. In reaching this decision,

students recognize that the sets included in these remainder systems are not only closed under addition and multiplication, but are also closed under the operation of subtraction.

Some special properties of the system of natural numbers are introduced in Lesson 12. These properties involve the natural numbers 0 and 1.

By using sets and the definition of sum, it is proved that the result of adding any natural number to 0 produces the natural number itself. Therefore, $0 + a = a$ yields a true statement for each replacement of a by members of N. The number 0 is called the identity element for addition of natural numbers.

Through the use of sets and the definition of product, it is proved that the natural number 1 is the identity element for multiplication of natural numbers. The result of multiplying any natural number by 1 produces the natural number itself. Therefore, $1 \cdot a = a$ yields a true statement for each replacement of a by members of N.

The notion of an identity element for addition and multiplication is extended by inspecting the finite sets introduced in Lesson 11. By examining the statements obtained from the addition and multiplication facts for these sets, students determine the identity elements for addition and multiplication of members of these sets.

Sets and the definition of product are again used to develop the zero property for multiplication of natural numbers. The result of multiplying any natural number by the number 0 is the number 0. The role of zero relative to the operations of subtraction and division of natural numbers is also considered in this lesson.

The content of the last two lessons of BOOKLET 7 comes under the heading of number theory. Here the student learns about relations between natural numbers. The first relation is "is a factor of." The universe for the variables is $\{1, 2, 3, \cdots\}$. If a true statement is obtained from $a \times b = c$, then the replacement for a is a factor of the replacement for c. The replacement for b is also a factor of the replacement for c. Since $2 \times 3 = 6$ is a true statement, 2 is a factor of 6 and 3 is a factor of 6. Notice that the relation "is a factor of"

is reflexive and transitive, but this relation is not symmetric. Thus, 2 is a factor of 2 is a true statement because $2 \times 1 = 2$ is a true statement. Also, since 2 is a factor of 6 and 6 is a factor of 12 are true statements, 2 is a factor of 12 is also a true statement. 2 is a factor of 6 is a true statement, but 6 is a factor of 2 is a false statement.

Using the idea of factor, the student constructs the Sieve of Eratosthenes for natural numbers from 1 through 100. He does this by systematically crossing out the names of numbers that have numbers other than one and themselves as factors. He also crosses out the number 1. The numbers whose names are not crossed out are the natural numbers from 1 through 100 that have exactly two factors, 1 and that number. These numbers are prime numbers. 2, 3, 5, 7, and 11 are the least five prime numbers.

There are many interesting questions about prime numbers. Some of these questions remain unanswered. Is there a formula for generating prime numbers? No one has been able to discover one so far. Is there a greatest prime number? Euclid proved that the set of prime numbers is an infinite set. Is there always a prime number between any natural number greater than 1 and two times that number? Mathematicians have proven that there is always at least one prime number between any natural number greater than 1 and two times that number.

Natural numbers that have more than two factors are <u>composite</u> numbers. 4, 6, 8, 9, and 10 are the least five composite numbers. Thus, the set of natural numbers greater than 0 can be separated into three subsets, namely, {1}, {prime numbers}, and {composite numbers}.

The student learns about the unique factorization theorem of arithmetic. When a number is expressed as a product of prime numbers or powers of prime numbers, we have a complete factorization of that number. For example, $2\times2\times2\times3$, or $2^3\times3$, is a complete factorization of 24. $3\times3\times5\times5$, or $3^2\times5^2$, is a complete factorization of 225. A prime number is considered to be a complete factorization of itself. Thus, 2 is a complete factorization of 2, 3 is a complete factorization of 3, and so on for each prime number. The product 2×3 and the product 3×2 are considered to be the same complete factorization of the composite number 6. The unique factorization theorem states that any natural number greater than

1 can be expressed as a product of prime numbers in just one way except for the order of the prime numbers in the product.

The student uses the idea of prime factor to help find the greatest common factor of two or more numbers. For example, to find the greatest common factor of 20, 40, and 60, he first finds a complete factorization of each number.

$20 = 2^2 \times 5$. $40 = 2^3 \times 5$. $60 = 2^2 \times 3 \times 5$.

The prime number 5 is a common factor of 20, 40, and 60. The prime number 2 is also a common factor of 20, 40, and 60. However, the greatest power of 2 that is a common factor is the second power. So the student uses 2^2 and 5 to find the greatest common factor of the three numbers. $2^2 \times 5$, or 20, is the greatest common factor of 20, 40, and 60.

PART THREE: CHARTING THE COURSE

Lesson Organization of BOOKLET 7

BOOKLET 7 is a part of a sequential program in mathematics for grades 7 and 8. Before using this booklet, students should study either BOOKLETS 1 through 5 or BOOKLET 6 of the Scott, Foresman Mathematics Program. BOOKLET 6 is a "revised - review" of BOOKLETS 1 through 5 and has been prepared primarily for eighth-grade students whose study of mathematics in grade seven was traditional.

The lessons in BOOKLET 7 consist of explanatory material, developmental exercises, and "On Your Own" exercises. The explanatory material and the developmental exercises (labeled A, B, etc.) are designed for class discussion. In this part of each lesson the major concepts are developed through the inductive approach. The examples in the numbered displays are used in developing the concepts.

Some of the more difficult developmental exercises are designed to give the student deeper mathematical insights. Such exercises are preceded by a large question mark. You will find notes on these exercises in Part Four of this TEACHER'S GUIDE, along with answers to other developmental exercises.

The "On Your Own" exercises provide opportunities for individual work. It will be most profitable to discuss these after all the students have completed them. These exercises may be used for supervised study in class, or they may be assigned as homework.

If the suggested time allotment is completely followed, the lessons in BOOKLET 7 will constitute about five weeks of classwork, without allowance for review and testing. The teacher, however, should feel free to adjust this schedule to his own classroom situation.

LESSON 1 (one class session)	Mapping, one-to-one correspondence, equivalent sets
LESSON 2 (one class session)	Natural numbers as properties of sets; standard sets of natural numbers

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Here for your students' examination is Booklet 7 of the Scott, Foresman Experimental Mathematics Project --

THE SYSTEM OF NATURAL NUMBERS

Because you requested and received earlier this year the first five booklets of the SF Experimental Mathematics Project for junior-high school, we thought you and your student teachers might also like to see this latest material now being tried out. (We have omitted sending you Booklet 6 because it is a condensation of the work in the first five booklets for use by eighth-graders new to the project this year.)

Booklet 7 answers the question, "What is a number system?" and clears up the differences between the idea of a numeration system (presented in Booklet 5) and that of a number system. Set language is used to define the natural numbers and the operations and properties of addition and multiplication. This booklet can show you and your student teachers how seventh-graders today are acquiring these modern mathematical ideas through a method of controlled discovery. The experimental material will be revised and published with other material in textbook form for use in Grade 7 next September as Seeing Through Mathematics 1, by Hartung, Van Engen, Trimble, Berger and Cleveland.

We hope you will find "The System of Natural Numbers" useful in giving your student teachers an idea of the new direction that junior-high school mathematics is taking.

Cordially yours,

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Teacher Education Service Division

LESSON 3 (two class sessions)	Ordering of natural numbers; ordering as reflected on a number line; betweenness and successor properties
LESSON 4 (two class sessions)	Union of sets; sum of two natural numbers; difference of two natural numbers; closure property of addition for natural numbers
LESSON 5 (two class sessions)	Definition of product of two natural numbers; quotient of two natural numbers; closure property of multiplication
LESSON 6 (two class sessions)	Commutative properties of addition and multiplication for natural numbers
LESSON 7 (two class sessions)	Associative properties of addition and multiplication for natural numbers
LESSON 8 (three class sessions)	Distributive property of multiplication over addition (natural numbers)
LESSON 9 (one class session)	Expressions, standard names, simplifying expressions, order of operations
LESSON 10 (one class session)	The system of natural numbers
LESSON 11 (two class sessions)	Finite number systems
LESSON 12 (two class sessions)	The identity elements for addition and multiplication
LESSON 13 (two class sessions)	Composite numbers, odd and even numbers, prime numbers, twin primes
LESSON 14 (one class session)	Complete factorization; unique factorization property; common factors; greatest common factors

PART FOUR: LESSON BRIEFS

Part Four provides background material and suggestions for teaching each lesson. The first part of each lesson brief contains notes that are intended to supplement the material given in Part Two of this TEACHER'S GUIDE. As you use the material for each lesson brief, you may wish to refer to the appropriate material in Part Two.

Answers for selected developmental exercises and all the answers for the "On Your Own" exercises are included in each lesson brief. Some answers to developmental exercises are obvious; thus, these answers are omitted in the lesson brief.

Be sure to observe the note at the bottom of the first page of each lesson. This note gives a technical description of the lesson content.

Some of the developmental exercises ask the student to explain his answer. These explanations should be given orally. However, the student should not be expected to give as precise an answer as is provided in the lesson brief. The answer given in the lesson brief should serve as a guide for discussion.

Use as many or as few of the developmental exercises as you consider necessary for your class. However,it is important to remember that the student will have a better understanding of a new concept through a slow and careful development.

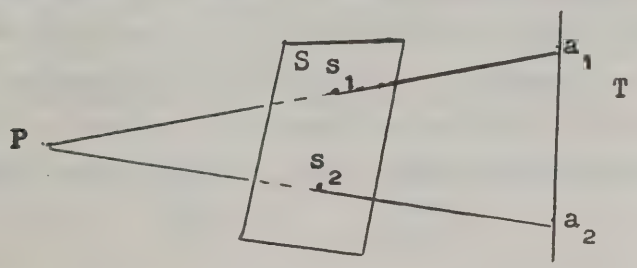
LESSON 1

Pages 1-5
Notes

The idea of a one-to-one correspondence between two sets is an important mathematical concept. The ideas of natural numbers and ordering of the natural numbers depend on it. The idea of one-to-one correspondence is intuitively understood so easily that there is a strong tendency to overlook details that are important, especially as a part of the student's foundation for the further study of mathematics.

To make possible a definition of one-to-one correspondence that is clear, accurate, and complete, we consider the idea of mapping. A mapping is a special type of pairing of the members of one set with the members of another set.

The word map, of course, is familiar to all students. On a geographical map each point is uniquely paired with a specific point on the surface of the earth. The process of making a map may be described as follows.



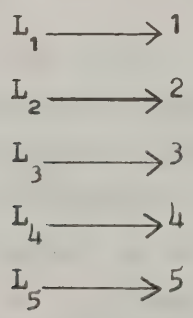
In the diagram above, point P represents the source of light from which light rays pass through a filmstrip S to a screen T. We can say that a typical point s_1 (read "s sub 1") is pictured on the screen at a_1. All the points s_1, s_2, s_3, ... on the filmstrip are mapped onto points a_1, a_2, a_3, ... on the screen T, and these points form a projection, or map, of the points on the filmstrip. We will return to this example later after we have seen how the idea of mapping has been extended by mathematicians.

If any two sets A and B are given, and if we associate each member of A with a member of B, and if each member of B is used, we have a mapping of A onto B. In Display 1, page 1, of the lesson we have a two-to-one mapping of a set of tickets onto a set of persons. We could have chosen an example in which the mapping is three-to-one. Whenever more than one member of a given set is mapped onto a member of a second set, we refer to the mapping as many-to-one.

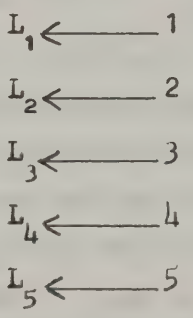
In the mapping illustrated by Display 2 each member of set C has an image in set D, and each member of D is used just once. This is a one-to-one mapping of set C onto set D. In a many-to-one mapping, a member of the second set is used more than one time.

Situations can arise in which two or more members of set A are mapped onto the same member in set B when A is mapped onto B. Consider, for instance, an overcrowded school in which theoretically each locker is "mapped onto" a student. If the available lockers run out before all the students receive an assignment, some of the lockers must be mapped onto two students. Note that each member of the set of lockers is mapped onto at least one member of the set of students. Also each student is used in the assignments (mapping) of the lockers to the students. This is a one-to-many mapping.

Now suppose that, before school opens, a clerk is engaged in assigning a number to each locker. The mapping that results is a one-to-one mapping of the set of numbers onto the set of lockers. It automatically determines another mapping. The second mapping is a one-to-one mapping of the set of lockers onto the set of numbers. The two mappings are partially shown in the diagrams below.



Mapping of lockers onto numbers



Mapping of numbers onto lockers

These two mappings show that there is a one-to-one correspondence between the set of lockers and the set of numbers. Such a correspondence between two sets can be expressed by using double-headed arrows to show the pairing of their members as follows.

$L_1 \longleftrightarrow 1$

$L_2 \longleftrightarrow 2$

$L_3 \longleftrightarrow 3$

$L_4 \longleftrightarrow 4$

$L_5 \longleftrightarrow 5$

One-to-one correspondence

Note especially that in a one-to-one correspondence the image of a particular member of either set is uniquely determined by either of the mappings from which it is structured. For example, 1 is mapped onto L_1 and L_1 is mapped onto 1.

The concept of one-to-one correspondence is used in defining equivalent sets. Equivalent sets are sets that can be put in one-to-one correspondence. If two equivalent sets are given, there are usually several one-to-one correspondences between the two sets. If "M = {a,b}" and "N = {c,d}," we have two possible one-to-one correspondences. They are shown below.

$$\begin{matrix}\{a,b\}\\ \updownarrow\updownarrow\\ \{c,d\}\end{matrix} \qquad \begin{matrix}\{b,a\}\\ \updownarrow\updownarrow\\ \{c,d\}\end{matrix}$$

$\begin{matrix}\{a,b\}\\ \updownarrow\updownarrow\\ \{d,c\}\end{matrix}$ shows the same correspondence as $\begin{matrix}\{b,a\}\\ \updownarrow\updownarrow\\ \{c,d\}\end{matrix}$ since in each case the same members of M and N are paired with each other.

Six different one-to-one correspondences can be set up between the sets {a,b,c} and {x,y,z}. The abler members of your class might find it interesting to to diagram these correspondences. Possibly some of them will deduce that there are 24 such correspondences for {a,b,c,d} and {w,x,y,z}. For two sets of $\underline{n}$ objects each, there are $1\times2\times3\times \ldots \times\underline{n}$ possible one-to-one correspondences.

The idea of many one-to-one correspondences between two given sets that are equivalent becomes tangible if we consider a one-to-one correspondence between a set of students and a set of lockers. Suppose the mappings assign student a_1 to locker L_{50}. If he exchanges lockers with student a_{30}, whose original assignment

was L_{24}, the original one-to-one correspondence is changed to a new one-to-one correspondence between the set of students and the set of lockers. In the new mappings, a_1 would be paired with L_{24} and a_{30} would be paired with L_{50}.

It is not possible to determine directly whether or not two infinite sets can be put in one-to-one correspondence. Mathematicians do so indirectly. They find a rule whereby each member of one infinite set can be paired uniquely with a member of another infinite set. In the example in this lesson brief, $\{s_1, s_2, s_3, \cdots\}$ and $\{a_1, a_2, a_3, \cdots\}$ are infinite sets. They can be put in one-to-one correspondence by using the rule that the point on the screen along the same light ray as a point on the filmstrip are images of each other. Thus, s_1 is paired with a_1, and a_1 is paired with s_1 in the two mappings of each set onto the other. Therefore, the two sets can be put in one-to-one correspondence, so they are equivalent.

As a second example, consider the familiar sets

$$C = \{1, 2, 3, 4, \cdots\} \quad \text{and}$$
$$D = \{1, 4, 9, 16, \cdots\}.$$

To map C onto D we find the second power or square of each member of C. The number 35 in C would be mapped onto 35^2, or 1225, in D. To map D onto C we find one of the two equal factors of each member of D. Thus, 49 in D would be mapped onto 7 in C, since "$7 \times 7 = 49$." Since the rule stated enables us to put the infinite sets C and D in one-to-one correspondence, we know that these sets are equivalent.

In teaching this lesson, the process of counting should not be used in any way to determine the equivalence of sets, because the idea of the numbers and the possibility of counting depend on one-to-one correspondences and equivalent sets. Subsequent lessons will show how natural numbers are associated with equivalent sets, and how this leads to the ordering of the natural numbers.

Use the displays in the lesson to explain many-to-one mappings, one-to-one mappings, and how two particular mappings form a one-to-one correspondence. Make sure that the students understand that in the mapping of one set onto another, each member of both sets must be included.

There are many examples of mappings and one-to-one correspondences in the experience of the students. They may be able to suggest some. To start a

discussion you might ask how one can determine whether or not there are enough textbooks for students without counting either textbooks or students. Or how they can show, without counting, that they have the same number of fingers on each hand.

Page 1

Answers for Exercise A — Two tickets

Exercise B — Yes, each member of A is matched with just one member of B. Each of the first two tickets are matched with Mary; each of the last two are matched with Steve.
Yes, each member of B is used in the matching.

Exercise C — One ticket

Exercise E — Two; two; yes, two tickets are mapped onto a single student.

Exercise F — One; one; one; one

Exercise G — Exactly one ticket is mapped onto each student. If the mapping were many-to-one, more than one ticket would be mapped onto at least one student.

Exercise H — The set of tickets that Ed sold

Page 2

Answers for Exercise I — One
There is an arrow from a member of set D pointing to each member of set C, so each member of C is used in the mapping.
The mapping is not a many-to-one mapping, since two or more members of set D are not mapped onto the same member of set C.
It is a one-to-one mapping because each member of D is mapped onto one member of C and all the members of C are used just once.

Exercise L — They are one-to-one mappings, because in each case exactly one member of a set has been mapped onto exactly one member of another set and every member of both sets has been used in the mappings.

Exercise M — There is an arrow from each member of set K pointing to exactly one unique member of set L. All the members of set L are used in the mapping.

Page 2

Answers for Exercise M (cont.) There would be an arrow from each member of set L pointing to exactly one unique member of set K, and all the members of set K would be used in the mapping. So the new mapping is a one-to-one mapping of set L onto set K.

Exercise N Yes; we know that there is a one-to-one mapping of set A onto set B. So we know that
(1) each member of A is matched with one member of B, and
(2) each member of B is used just once in the matching.

We can now obtain a one-to-one mapping of set B onto set A. We must first show a way to map each member of B onto one member of A.

For each member of A that was mapped onto a member of B in the first mapping, we now map that member of B onto that member of A.

We must also show that each member of A will be used just once.

From the first mapping, we know that each member of B was used just once, so we now know that two different members of B will not be mapped onto the same member of A. (The new mapping will not be a many-to-one mapping.) A member of set A will not be used more than once. We also know that in the first mapping, each member of A was used, so in the new mapping each member of A will also be used. Each member of A will be used once.

In the new mapping, each member of B will be mapped onto just one member of A, and each member of A will be used just once. So the mapping of set B onto set A will also be a one-to-one mapping.

Exercise O Yes, because there is a one-to-one mapping of the set of cups onto the set of saucers, and a one-to-one mapping of the set of saucers onto the set of cups.

Page 3

Answers for Exercise P The second baseball; the second bat

Exercise Q Yes, there is a one-to-one mapping of set L onto set K and there is a one-to-one mapping of set K onto set L.

Exercise A 6 is mapped onto a, 5 is mapped onto b, 9 is mapped onto c, 3 is mapped onto d, and 1 is mapped onto e.

a is mapped onto 6, b is mapped onto 5, c is mapped onto 9, d is mapped onto 3, and e is mapped onto 1.

Sets A and B are in one-to-one correspondence, since there is a one-to-one mapping of set A onto set B and there is a one-to-one mapping of set B onto set A.

Page 3 (cont.)

Answers for Exercise B

9 is mapped onto a, 3 is mapped onto b, 6 is mapped onto c, 5 is mapped onto d, and 1 is mapped onto e.

a is mapped onto 9, b is mapped onto 3, c is mapped onto 6, d is mapped onto 5, and e is mapped onto 1.

Yes, because the two one-to-one mappings shown in Display 9 are different from the one-to-one mappings shown in Display 8.

Exercise C

Often two sets can be put in one-to-one correspondence in several ways. The number of ways in which this can be done depends on the number of members in each of the sets. For example, you will see in Exercise F that two sets with three members each can be put in one-to-one correspondence in 6 different ways.

These two sets can be put in one-to-one correspondence in 120 different ways. Here are two samples.

6	1	5	3	9
a	b	c	d	e

1	3	5	9	6
b	d	a	e	c

Exercise D

Yes; no

Sets A and C are not in one-to-one correspondence because there is a member of C not used in the mappings.

Exercise E

The two sets can be put in one-to-one correspondence in 120 different ways. Two possibilities are shown below.

6	5	9	3	1
chair	table	apple	pear	orange

6	1	3	5	9
chair	table	apple	pear	orange

Exercise F

Six

Note: The number of ways to do this for two sets of n objects each is the same as the number of possible arrangements of n objects. n objects can be arranged in $1 \times 2 \times 3 \times \ldots \times n$ possible ways. For 3 objects, there are $1 \times 2 \times 3$, or 6, possible ways.

Exercise G

F and H are infinite sets. This is indicated by the sequence of three dots included in the tabulation of set F and of set H.

Page 3 (cont.)

Answers for Exercise H — Sets D and G

Exercise I — Sets D and G, because these sets can be put in one-to-one correspondence with {6, 5, 9, 3, 1}

Page 4

Answers for Exercise J — No, because it is possible to tell how many members the set has

No, because it cannot be put in one-to-one correspondence with an infinite set

Exercise K — R is the set of natural numbers from 0 through 4.
S is the set of even numbers from 0 through 8.

Exercise L — The products are 0, 2, 4, 6, 8. Each of these products is a member of S.

Exercise M — Yes, the arrows pointing downward show that each member of R is mapped onto a member of S that is two times that member of R.

A mapping of S onto R is shown by the arrows pointing upward.

Exercise N — Display 13 shows a one-to-one correspondence between sets R and S. Since they can be put in one-to-one correspondence, they are equivalent sets.

Exercise A — Set N is the set of natural numbers. Set E is the set of even natural numbers.

These sets are infinite sets because it is not possible to tell how many members each set has.

Exercise B — 3 is a member of N, and 3×2, or 6, is a member of E.
17 is a member of N, and 17×2, or 34, is a member of E.
28 is a member of N, and 28×2, or 56, is a member of E.

Exercise C — Yes, because the product of 2 and any natural number is an even number; 5 is mapped onto 2×5, or 10; 25 is mapped onto 2×25, or 50; 500 is mapped onto 2×500, or 1000; 10,001 is mapped onto $2 \times 10{,}001$, or 20,002.

Exercise D — Yes, each member of N can be mapped onto 2 times itself, and the product will be a member of E. The product will be a different member of E for each member of N that we use. So each member of N can be mapped onto a different member of E.

Page 4

Answers for Exercise D (cont.) Yes, each member of E will be used, because the products obtained by multiplying each natural number and 2 are the only even numbers.

Exercise E The number that $2\underline{x}$ will express after replacement

Exercise H 35 is not a member of E, since 35 is not an even number.

Exercise I Divide each member of E by 2.
4; 9; 403

Exercise J Yes; for example, we would map 4,234 onto 4,234 ÷ 2, or 2,117.

Exercise K The number that $\underline{x} \div 2$ will express after replacement

Page 5

Answers for Exercise L We have put the set of even numbers and the set of natural numbers in one-to-one correspondence, since there is a one-to-one mapping of the set of natural numbers onto the set of even numbers and there is a one-to-one mapping of the set of even numbers onto the set of natural numbers. We used a rule to map the members of each set onto the members of the other set.

Exercise M Yes, because they can be put in one-to-one correspondence

Exercise N

$$N = \{0 \quad 1 \quad 2 \quad 3 \quad 4 \quad \ldots\}.$$
$$\updownarrow \quad \updownarrow \quad \updownarrow \quad \updownarrow \quad \updownarrow$$
$$O = \{1 \quad 3 \quad 5 \quad 7 \quad 9 \quad \ldots\}.$$

If $\underline{x}$ is a variable for any member from N, $\underline{x}$ is mapped onto $2x + 1$ in O. If $\underline{x}$ is a variable for any number from O, $\underline{x}$ is mapped onto $(x - 1) \div 2$ in N.

Exercise O Yes

Consider the following diagram:

$$\begin{array}{ccccccc} 0 & 2 & 4 & 6 & 8 & \ldots \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \\ 1 & 3 & 5 & 7 & 9 & \ldots \end{array}$$

It suggests two rules to map the members of each set onto the members of the other set. The rules are:

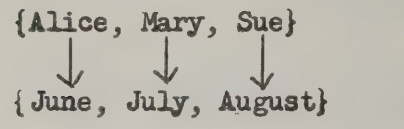
1 Add one to each member of the set of even numbers to find the member of the set of odd numbers onto which it will be mapped.

Page 5
Answer for Exercise 0 (cont.)

2 Subtract one from each member of the set of odd numbers to find the member of the set of even numbers onto which it will be mapped.

Answers for "On Your Own" exercises

Note: There is more than one correct answer to each of the exercises from 1 through 5. In each case one possible answer is given.

1


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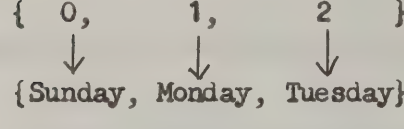

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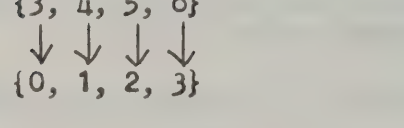

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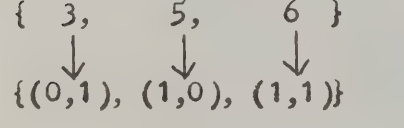

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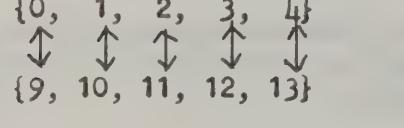

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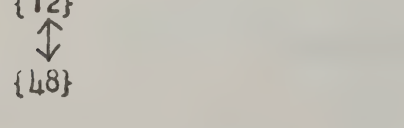

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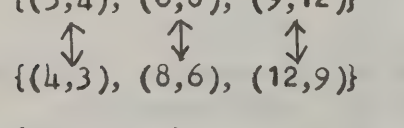

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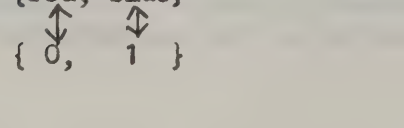

10

11

12

13

14

15

Note: This correspondence can be continued without end. To find the number in the first set to be mapped onto the successor of 5, multiply 100,000 by 10. Another way to find this number is to think of the first set as powers of ten. Then 1 is mapped onto the first power of ten, 2 is mapped onto the second power of ten, ... 6 is mapped onto the sixth power of ten, and so on.

LESSON 2

Pages 6-10
Notes

This lesson shows how the natural numbers emerge in a special way from equivalent sets. We shall now consider sets that are standard sets, and we begin with the empty set symbolized { }. We associate the natural number zero with the empty set. The sentence "0 = n{ }" should be read, "The natural number 0 equals the number of members in the empty set."

The standard set that has a single natural number as a member is {0}. The sentence "1 = n{0}" should be read, "The natural number 1 equals the number of members in the standard set whose member is the natural number 0."

Now we have some natural numbers to work with. They are 0 and 1. So we can make another standard set. Its members are 0 and 1. The standard set is {0,1}. Like { } and {0}, the standard set {0,1} is associated with a natural number. This number is 2. The sentence "2 = n{0,1}" is read, "The natural number 2 equals the number of members in the standard set whose members are 0 and 1." Now we have the natural numbers 0, 1, and 2. The number 0 is the common property of the empty set and all sets equivalent to the empty set. The number 1 is the common property of {0} and all sets equivalent to {0}. The number 2 is the common property of {0,1} and all sets equivalent to {0,1}. One such set is {father, mother}. Another is {//}. A third is {table, chair}.

The natural numbers 0, 1, 2 are the members of the standard set {0,1,2}. A new natural number is needed to describe the common property of this set and all sets equivalent to it. This natural number is 3, and "3 = n{0,1,2}" expresses a true statement.

The developmental exercises in the lesson help the student discover how each standard set is constructed out of the natural numbers previously defined. Finally, we generalize by saying that, if m is replaced by any member of {1,2,3, ···}, then "m = n{0,1,2, ···, m - 1}" will be a true statement.

A very important point for the student to realize is that a natural number is the common property of a standard set and all sets to which it is equivalent. For example, the natural number 4 is the common property of {*, ≠, ?, ◊} and the standard set {0, 1, 2, 3}. These sets have a common property

because they are equivalent sets. They are equivalent because they can be put in one-to-one correspondence. The set {quart, pint, gallon, liter}, as well as many other sets, has this same common property, because it is also equivalent to {0, 1, 2, 3}. Moreover, a set that is not equivalent to {0, 1, 2, 3} does not have this property. So the standard set {0, 1, 2, 3} and sets equivalent to {0, 1, 2, 3} are the only sets associated with 4.

Comparable observations can be made concerning any other member of the set of natural numbers. Each is the common property of a standard set and all sets equivalent to this standard set.

To teach the lesson, it would probably be well to consider all the developmental exercises in a class discussion with all the students participating.

Page 6

Answers for Exercise A

A set of five boats	A set of three automobiles
A set of three airplanes	A set of five flags
A set of one telephone	A set of seven books
A set of seven pencils	A set of five bicycles

Exercise B

The set of boats can be mapped onto the set of flags by associating each boat with a single flag. The mapping is one-to-one because each flag is used just once.

The set of flags can be mapped onto the set of boats by associating each flag with a single boat. This mapping is also one-to-one because each boat is used just once.

Exercise C

The set of boats and the set of flags can be put in one-to-one correspondence as follows.

Make a one-to-one mapping of the set of boats onto the set of flags. This mapping automatically determines a second mapping by linking a specific flag with a specific boat. Without changing the pairs, map the set of flags onto the set of boats. The result is a one-to-one correspondence between the two sets. So the set of boats is equivalent to the set of flags.

Exercise D

The set of bicycles

A one-to-one correspondence can be set up between the set of boats and the set of bicycles by the method used to put the set of boats and the set of flags in one-to-one correspondence.

Since the two sets can be put in one-to-one correspondence, they are equivalent sets.

Page 6 (cont.)

Answers for Exercise F — These sets have the same number of members because any pair of them can be put in one-to-one correspondence.

Exercise G — {sea, ocean, river, lake, stream}, {sweater, hat, coat, dress, blouse}, and {2, 4, 6, 8, 10} are each equivalent to the set of boats, the set of flags, and the set of bicycles.

Exercise H — No, because if we try to map the set of airplanes onto the set of boats, the mapping could not be one-to-one. Remember that all the airplanes and all the boats must be included in the mapping.

Exercise I — The set of automobiles

Yes, the set of airplanes and the set of automobiles are equivalent because they can be put in one-to-one correspondence.

Exercise J — The set of airplanes and the set of automobiles can be put in one-to-one correspondence, so they have the same number of members.

Exercise K — The set of airplanes and {book a, book b, book c} are equivalent sets.

Exercise L — Any two of the three sets can be put in one-to-one correspondence. Therefore, any two of the sets have the same number of members.

Exercise P — No other set pictured in the display, since the set whose member is a telephone cannot be put in one-to-one correspondence with any of the other sets.

Exercise Q — Yes, because {Illinois} can be put in one-to-one correspondence with a set whose member is a telephone

Page 7

Answers for Exercise A — No members; the empty set

Exercise B — The set of palm trees at the North Pole is one possible answer.

Exercise C — Yes, because each set is the empty set; because there are no members in either of the sets mentioned

Exercise D — The set whose member is the number zero

Exercise E — { } is a set that has no members, while {0} is a set that has one member, the number 0.

Page 7

Answers for Exercise E (cont.) n{ } is the number of members in the empty set, so n{ } is zero. {0} is the set whose one member is the number zero. n{ } refers to a number; {0} refers to a set.

Exercise F
A is the set whose member is Canada.
B is the set whose member is quarterback.
C is the set whose member is a.

Exercise G All of the sets mentioned are equivalent because each of them can be put in one-to-one correspondence with any one of the others.

Exercise H {house}, {pencil}, and {mathematics} are equivalent to each of the sets named in Displays 3 and 4.

Exercise I
The natural number 1 equals the number of members in the set whose member is zero.

The natural number 1 equals the number of members in the set whose member is Canada.

The natural number 1 equals the number of members in the set whose member is quarterback.

The natural number 1 equals the number of members in the set whose member is a.

Page 8

Answers for Exercise J 1, because the set tabulated can be put in one-to-one correspondence with {0}

Exercise K The natural number 1

Exercise L The set whose members are 0 and 1

Exercise M
The set whose members are France and England
The set whose members are a and b
The set whose members are hat and coat

Exercise N Yes, because each of the sets tabulated can be put in one-to-one correspondence with {0,1}

Exercise P
The natural number 2 equals the number of members in the set whose members are 0 and 1.

Yes, because each of these sets is equivalent to {0,1}

Exercise Q The natural number 2

Exercise R
0 = n{ }.
1 = n{0}.
2 = n{0,1}.

Page 8 (cont.)

Answers for Exercise A — The set whose members are 0, 1, and 2

Exercise B — The set whose members are Ohio, Iowa, and Texas
The set whose members are trumpet, drum, and clarinet

Set A is equivalent to set B because these sets can be put in one-to-one correspondence.

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Answers for Exercise C — Each set is equivalent to {0, 1, 2} because each set can be put in one-to-one correspondence with {0, 1, 2}.

Exercise D — The natural number 3 equals the number of members in the set whose members are 0, 1, and 2.

{0, 1, 2} is the standard set of 3, because it has as its members natural numbers. Its members are just those numbers that have been associated with standard sets so far.

Exercise E — 3

The sets in Display 11 are each equivalent to the standard set {0, 1, 2}. So the common property of these sets and {0, 1, 2} is the natural number 3.

Exercise F — The natural number 3

Exercise G — The natural number 4 equals the number of members in the set whose members are 0, 1, 2, and 3.

Yes, because {0, 1, 2, 3} contains only natural numbers, and it contains all the natural numbers so far developed.

Exercise H — There are many correct answers. The following is typical.
{John, Henry, Harry, Bill}

{sun, rain, wind, storm}
The natural number 4 is the common property.

Exercise I — The natural number 4

Exercise J — The natural number 4

Exercise K

A:	0	E:	4
B:	1	F:	5
C:	2	G:	6
D:	3		

Exercise L — {Sunday, Monday, Tuesday, Wednesday, Thursday, Friday, Saturday}

Page 9 (cont.)

Answers for Exercise M Yes, because {0,1,2,3,4,5,6} can be put in one-to-one correspondence with the set whose members are the days of the week

Exercise N The natural number 7

Exercise O The standard set of 12

It contains only natural numbers as members, and its members are exactly those natural numbers developed by the method explained in the lesson that make up a set equivalent to all sets whose common property is the natural number twelve.

Exercise P {0,1,2,3,4,5,6,7}

{0,1,2,3,4,5,6,7,8}

{0,1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18}

{0,1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19}

Exercise Q The standard set of $\underline{m}$ when $\underline{m}$ is replaced by any natural number other than 0 is $\{0,1,2,3, \cdots, m-1\}$.

When 0 is the replacement for $\underline{m}$, the standard set is $\{\ \}$.

The standard set of $\underline{m}$ is the set of natural numbers whose members are all the natural numbers associated with standard sets so far. For example, if $\underline{m}$ is replaced by the number 97, the standard set would be $\{0,1,2, \cdots, 95, 96\}$.

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Answers for Exercise S N

Yes, because it has an unlimited number of members

Exercise T No; if $\underline{m}$ is a variable for any member of N, the standard set of $\underline{m}$ has exactly $\underline{m}$ members. Since we can count the members, the standard set is not an infinite set.

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Answers for "On Your Own" exercises

Note: There are many correct answers for Exercises 1 through 5. Those given are typical.

1 {Mississippi, Amazon, Danube, Nile, St. Lawrence}

2 {potato, carrot, beet, cabbage, bean}

3 {bed, chair, table, lamp, desk}

Answers for "On Your Own" exercises (cont.)

4 {robin, thrush, bluejay, sparrow, wren}

5 {football, baseball, basketball, hockey, tennis}

6 The natural number 5

7 13 = n{0,1,2,3,4,5,6,7,8,9,10,11,12}.

8 10 = n{0,1,2,3,4,5,6,7,8,9}.

9 0 = n{ }.

10 11 = n{0,1,2,3,4,5,6,7,8,9,10}.

11 15 = n{0,1,2,3,4,5,6,7,8,9,10,11,12,13,14}.

12 21 = n{0,1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20}.

13 {0,1,2,3}

14 {0,1,2,3,4,5}

15 {0,1}

16 {0}

17 {0,1,2}

18 {0,1,2,3,4}

19 {0,1,2,3,4,5,6}

20 { }

21 {0,1,2,3,4,5,6,7,8,9,10,11,12,13}

22 {0,1,2,3,4,5,6,7,8,9,10,11,12,13,14}

23 11 = n{0,1,2,3,4,5,6,7,8,9,10}.

24 14 = n{0,1,2,3,4,5,6,7,8,9,10,11,12,13}.

25 18 = n{0,1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17}.

26 16 = n{0,1,2,3,4,5,6,7,8,9,10,11,12,13,14,15}.

27 15 = n{0,1,2,3,4,5,6,7,8,9,10,11,12,13,14}.

Answers for "On Your Own" exercises (cont.)

28 $9 = n\{0,1,2,3,4,5,6,7,8\}$.

29 $1 = n\{0\}$.

30 $17 = n\{0,1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16\}$.

31 $13 = n\{0,1,2,3,4,5,6,7,8,9,10,11,12\}$.

32 $\underline{m} = n\{0,1,2,3, \cdots, \underline{m} - 1\}$.

33 No, because $\{0,1,2,3,4\}$ cannot be put in one-to-one correspondence with $\{0,1,2,3,4,5\}$

LESSON 3

Pages 11-15 Notes

In this lesson the student learns about some of the properties of natural numbers. One of these is the order property. The order relations that are relevant to the natural numbers are "is less than" and "is greater than." We can select either of these as a point of departure, since for every replacement of the variables, when the universe is N, the statement obtained from "$a < b$" gives us the same information as the statement obtained from "$b > a$." We will use "is less than" in our discussion.

To order the natural numbers we must establish criteria for determining (1) whether or not $\underline{a}$ is less than $\underline{b}$ for any given replacement for the variables, (2) given $\underline{a}$, what number should immediately follow $\underline{a}$, and (3) which member of N shall be first in the ordered set. We cannot use counting to order the natural numbers. The process used to count is based on the assumption that the natural numbers have already been ordered. To order the natural numbers we will use the concept of standard set and the concept of proper subset. The student has already learned that each natural number is associated with a standard set. The concept of proper subset is a simple extension of the concept of subset. When a subset does not contain all the members of a set, that subset is a proper subset.

To order any two natural numbers, we will use the concept of standard set and the concept of proper subset. Consider the number 4 and the number 7. The

standard set of 4 is {0, 1, 2, 3} and the standard set of 7 is {0, 1, 2, 3, 4, 5, 6}. Since the standard set of 4 is a proper subset of the standard set of 7, the number 4 is less than the number 7. In general, when the standard set of one number is a proper subset of the standard set of another number, the first number is less than the second number.

Now we will determine which number immediately follows 4. Each member of the standard set of 4 is less than 4. Consider the set that contains only these numbers and 4. This is {0, 1, 2, 3, 4} or the standard set of 5. So we select 5 as the number that will immediately follow 4, and we refer to 5 as the successor of 4. In the same way, we can identify the successor of any natural number, and we say that the natural numbers have the <u>successor property</u>.

In <u>Time out for a chat</u>, page 12, we ask the student to accept the observation that 0 is less than each of the other natural numbers. The reason for this is that the concept of the empty set as a proper subset is difficult for the students. However, some of the students may readily see that a logical way to include 0 is to think of { } as a proper subset of {0}, {0, 1}, and all the other standard sets for natural numbers. Also, the successor of 0 is the number whose standard set includes 0 and only those natural numbers less than 0. This standard set is {0}. So the successor of 0 is 1.

We can now arrange the natural numbers in a special order so that each natural number will be less than any other natural number following it. In this way, we obtain the ordered set of natural numbers. 0 is first, since 0 is the least natural number; then comes 1, 2, 3, and so on.

There are several interesting and important observations that we can make about the ordered set of natural numbers. One is that every member of the set has exactly one successor, and if a is a variable for a member of the set, then $a + 1$ is a description of the successor of a. Another is that, since "U = N," 0 is not the successor of a member of N. In a different universe we could think of 0 as the successor of -1, but -1 is not a member of the set of natural numbers. A third observation is that, given a set of natural numbers with the properties (1) 0 is a member of this set and (2) if x belongs to this set, the successor

of x also belongs to it, we know that this set contains all the natural numbers.

After we have ordered the set of natural numbers, another property emerges. This property is that of betweenness. It is possible to determine how many numbers are between any two natural numbers in the ordered set N and what these numbers are by using a compound condition and finding its solution set. If a and b are variables for two members of N and x is a variable for a member between a and b, the betweenness property states "$a < x \wedge x < b$." So to find the members of N between a and b, we determine the solution set of this condition. Consider an example. Select any two members of N as replacements for a and b respectively. The only restriction is that a must be less than b. If we replace a by 11 and b by 15, the condition becomes "$11 < x \wedge x < 15$." The solution set of this condition is {12, 13, 14}. We know then that there are three numbers between 11 and 15 in the ordered set of natural numbers and that these numbers are 12, 13, and 14.

Some generalizations can be made about the betweenness property from the condition "$a < x \wedge x < b$." "$a < x < b$" is an abbreviated form for this condition. If $\{x \mid a < x < b\}$ is the empty set, b is the successor of a. If it has one member, there is one number between a and b in the ordered set N. In every case the number of members of N between a and b is finite, since $\{x \mid a < x < b\}$ is always a finite set.

It is interesting to note that the set of all fractions has a betweenness property, but it differs from that of the natural numbers. For any two fractions we can always find another fraction between them. Their average is one such fraction. Consequently, between any two fractions there is an infinite number of other fractions, and no fraction has an immediate successor. We will discuss these matters in detail in a subsequent booklet.

Once we have order property, we can assign numbers to points on a line, so that the successor of a number is assigned to a point immediately to the right of the point to which the number itself is assigned. The direction is arbitrary, but by custom we move to the right on the number line. For each point assigned to a, there is a unique point assigned to the successor of a. If the point

assigned to a is point a, then the point assigned to the successor of a is point a + 1. Dots picturing the points assigned to the natural numbers are equally spaced. The dot farthest to the left pictures the point assigned to 0.

Enough time should be spent on this lesson for thorough understanding. Those students who quickly master the content of the text could identify order relations other than "is less than," and apply the principles learned to the ordering of sets other than the natural numbers.

A set is ordered if for any two members, a and b, there is a way to determine that a precedes b or a follows b, but a cannot both follow and precede b. There are many examples of ordered sets. The days of the week, the notes of a musical scale, the descendants of a given progenitor, places on a longitudinal meridian ordered according to direction to the north of or to the south of a starting point are a few examples.

The students should choose examples of sets that can be ordered, and investigate to see if they exhibit the same properties or different properties from those of the ordered set of natural numbers.

The above-average student could also consider such questions as the following.

1 If a, b, and c are variables for members of the ordered set S of odd natural numbers, which of the following statements are true for every replacement of a, b, and c?

$a \prec a$.

If $a \prec b$ and $b \prec c$, then $a \prec c$.

If $a \prec b$, then $b \prec a$.

If $a \prec b$, then $b \succ a$.

If $a \prec b$, then the successor of $a \prec$ the successor of b.

If c is the successor of a, and c is the successor of b, then $a = b$.

Every member of S has a successor.

Every member of S is the successor of another member of S.

$a + 1$ is the successor of a.

S has a first member.

S has a last member.

2 If a set S has n members, S has how many subsets?

Note: The answer is 2^n if the empty set and the set itself are included.

If "S = {a, b, c}," it has 2^3 or 8 subsets. They are tabulated below.

{a,b,c}	{a,c}	{a}	{c}
{a,b}	{b,c}	{b}	{ }

Page 11

Answers for Exercise A Since each member of {cat,dog,lion} is a member of S, this set is a subset of S.

Exercise B No; it does not contain tiger.

Exercise C {cat,dog} is a subset of S, because each member of {cat,dog} is a member of S. {cat,dog} is a proper subset of S, because {cat,dog} does not contain all the members of S.

Exercise D All the other sets tabulated in Display 1 are proper subsets of S.

Exercise E Yes; yes; yes; no
S is a subset of itself because each member of S is contained in S.
S is not a proper subset of itself because a proper subset must not contain all the members of the set.

Exercise F Yes, because all the members of {0} are members of {0,1}, but not all the members of {0,1} are members of {0}.

Exercise G {0}, {2}, {0,2}, {1,2}
This is one answer. There are other possible answers.

Exercise H Yes. The standard set of 1 is {0}. The standard set of 2 is {0,1}. See Exercise F.

Exercise I Yes. The standard set of 2 is {0,1}. The standard set of 3 is {0,1,2}. All the members of {0,1} are members of {0,1,2}, but all the members of {0,1,2} are not members of {0,1} and, therefore, {0,1} is a proper subset of {0,1,2}.

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Answer for Exercise J The standard set of 1 is a proper subset of the standard set of 2.
The standard set of 2 is a proper subset of the standard set of 3.

Page 12 (cont.)

Answers for Exercise L $\{0,1,2,3\}$

The standard set of 3 is $\{0,1,2\}$. It is a proper subset of the standard set of 4.

Exercise M The number 3 is less than the number 4.

If the standard set of 3 is a proper subset of the standard set of 4, then we know that "$3 < 4$" is a true statement.

Exercise N The inequality "$4 > 3$" gives us the same information as "$3 < 4$."

Exercise O $\{0,1,2,3,4\}$

The standard set of 4 is $\{0,1,2,3\}$. This is a proper subset of the standard set of 5. So "$4 < 5$" is a true statement.

Exercise P $5 > 4$.

Exercise Q Yes. The standard set of 6 is $\{0,1,2,3,4,5\}$. The standard set of 5 is $\{0,1,2,3,4\}$. Since the latter is a proper subset of the former, we know that "$5 < 6$" is a true statement.

Exercise R $6 > 5$.

Exercise A The standard set of 7 is a proper subset of the standard set of 8.

The standard set of 8 is a proper subset of the standard set of 7.

A true statement is obtained from the first condition.

Exercise B The two statements are "$7 < 8$," and "$8 < 7$."

A true statement is obtained from the first condition.

Exercise C The standard set of 10 is a proper subset of the standard set of 9.

The standard set of 9 is a proper subset of the standard set of 10.

A true statement is obtained from the second condition.

Exercise D "$10 < 9$," and "$9 < 10$" are the two statements.

A true statement is obtained from the second condition.

Exercise E A true statement will be obtained from "$a < b$" whenever the standard set of the member of $\{1,2,3, \cdots\}$ selected

Page 12

Answer for Exercise E (cont.) as a replacement of a is a proper subset of the standard set of the member of {1,2,3, ···} selected as a replacement of b.

Page 13

Answers for Exercise F All the statements are true because 0 is less than any other natural number.

Exercise G {1,2,3,4, ···}

Exercise H Since {0} is a proper subset of {0,1}, "$1 < 2$" is a true statement.

Since {0} is a proper subset of {0,1,2}, "$1 < 3$" is a true statement.

Since {0} is a proper subset of {0,1,2,3}, "$1 < 4$" is a true statement.

Since {0} is a proper subset of {0,1,2,3,4}, "$1 < 5$" is a true statement.

Exercise I {2, 3, 4, 5, ···}

Exercise J Yes. Since the standard set of 1 is a proper subset of the standard set of any other natural number except 0, we know that the number 1 is less than each of the other natural numbers except 0.

Exercise K {3, 4, 5, 6, ···}

Yes. Since the standard set of 2 is a proper subset of the standard set of any other natural number except 0 and 1, we know that the number 2 is less than each of the other natural numbers except 0 and 1.

Exercise L Yes, because the standard set of each number is a proper subset of the standard set of each of the numbers after it

Exercise M 0; zero is less than each of the other natural numbers.

Exercise N Yes; see diagram in Display 5.

Yes, because the standard set of 1 is a proper subset of the standard set of each of the other natural numbers except zero

Exercise O Yes; see diagram in Display 5.

Yes, because the standard set of 2 is a proper subset of the standard set of each of the other natural numbers except 0 and 1

Page 13 (cont.)

Answers for Exercise P — Because 3 is less than each of the natural numbers except 0, 1, and 2. The standard set of 3 is a proper subset of the standard set of each of the other natural numbers except 0, 1, and 2.

Exercise Q — 7; 10
We agree that zero is less than each of the other natural numbers.

Exercise S — Point 1 is to the left of points assigned to numbers greater than 1.

Exercise T — Point 2 is to the left of points assigned to numbers greater than 2.

Exercise U — Point a is to the left of point b on the number line.

Exercise A — 8; 38; 1; 5001

Exercise B — Yes; add the number 1 to the given natural number.

Exercise C — 6; 1; 73,926
Each natural number has a successor. If a is a variable for a natural number, a + 1 is the successor of a for every replacement of the variable.

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Answers for Exercise D — No
If a is a natural number, its successor is a + 1.
If b is a natural number, its successor is b + 1.
"a + 1 = b + 1" will become a true statement only if the variables a and b are replaced by the same natural number.
Therefore, it is not possible for two different natural numbers to have the same successor.

Exercise E — No, because if such a number exists, it must be less than 0; and the agreement that 0 is less than any other natural number would be contradicted. Note: Negative numbers are not natural numbers.

Exercise F — Yes. If 0 is a member of the set and the set contains the successor of every member of the set, then the set must contain 1, because 1 is the successor of 0. If the set contains 1, then it must contain 2, because 2 is the successor of 1. If the set contains 2, then it must contain 3. If the set contains 3, then it must contain 4.

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Answers for Exercise F (cont.) Continuing in this way, we see that all the natural numbers will be members of the set.

Exercise G The points assigned to 6, 7, 8, and 9

The solution set of the condition "$5 < x$" is $\{6,7,8, \cdots\}$. The solution set of the condition "$x < 10$" is $\{0,1,2,3,4,5,6,7,8,9\}$. So the solution set of the compound condition "$5 < x \wedge x < 10$" is $\{6,7,8,9\}$.

Each of the numbers assigned to points between point 5 and point 10 satisfies the compound condition "$5 < x \wedge x < 10$" because each number must be greater than 5 but less than 10.

Exercise H $\{9, 10, 11, 12, 13, 14, 15\}$

No; it has exactly 7 members.

It is the solution set of the compound condition "$8 < x \wedge x < 16$."

Exercise I One possible answer is given below.

If the two numbers chosen are 35 and 38, the set is $\{36,37\}$. It is not an infinite set, and the compound condition "$35 < x \wedge x < 38$" is satisfied by the numbers 36 and 37 whose points are between the points assigned to 35 and 38.

Exercise J No, because the solution set of the compound condition "$7 < x \wedge x < 8$" in "$U = N$" is the empty set

Exercise K $\{\ \}$

Pages 14-15

Answers for "On Your Own" exercises

Note: The sets named in Exercise 1 have more than three proper subsets. One possible answer is given for each set.

1 a $\{0\}$ $\{1,2\}$ $\{0,2\}$

b $\{2,3,4\}$ $\{4,5\}$ $\{1\}$

c $\{(1,2), (1,4)\}$ $\{(1,3), (1,4)\}$ $\{(1,4)\}$

d $\{0,1\}$ $\{2,3,4,5\}$ $\{7,8,9\}$

e $\{a\}$ $\{a,b\}$ $\{b,c,d\}$

f $\{2,3\}$ $\{0,4,5\}$ $\{0,1,2,3,4\}$

Answers for "On Your Own" exercises (cont.)

2-7 In Exercises 2, 3, and 6, the standard set associated with the first component of the ordered pair is a proper subset of the standard set associated with the second component.

In Exercises 4, 5, and 7, the standard set associated with the first component of the ordered pair is not a proper subset of the standard set associated with the second component.

8 The ordered pairs named in Exercises 2, 3, and 6.

9 Yes; if a replacement for (x,y) satisfies "$x < y$," the replacement also satisfies "$y > x$," and conversely.

10 8

11 {10,11,12,13,14,15,16,17,18,19,20}

12 {1,2,3, ···,13}

13 { }

LESSON 4

Pages 15-23 Notes

The definition of the sum of two natural numbers involves the idea of disjoint sets. First let us see what is meant by disjoint sets. Suppose that, at Lincoln High School, some of the students are enrolled in Latin classes, and some are members of a varsity team. It is possible that students who elect Latin also belong to a varsity team, but equally possible that no students participate in both pursuits. In the latter case the two sets of students are said to be disjoint. To be thus classified, two sets must have no members in common. In this example, if some students belong to both sets, the number of different students involved cannot be found directly by the operation known as addition. The union of the two sets, whether or not they are disjoint, is the set that includes all the students who either belong to a Latin class or to a varsity team or both.

If A = {a,b,c}, and B = {d,e}, set A and set B are disjoint. The union of A and B, symbolized A $\cup$ B, is {a,b,c,d,e}. But if A = {a,b,c}, and C = {b,c,d,e,f}, set A and set C are not disjoint since b and c are members of both. A $\cup$ C = {a,b,c,d,e,f}.

Sometimes we refer to the num (pronounced n$\breve{\text{u}}$m) of a set A. It is the number associated with the set, and may be symbolized n(A). In the above example, 3 = n(A), 2 = n(B), and 5 = n(A $\cup$ B). But 5 = n(C), and 6 = n(A $\cup$ C). The sum of n(A) and n(B) is n(A) + n(B). n(A) + n(B) = n(A $\cup$ B) is a true statement only if A and B are disjoint sets. A definition of the sum of two natural numbers is given on page 18 of the student's text. When reading the definition, the student should understand that although we use s, we could use any letter to refer to the member of N that is associated with set S. He should further understand that this statement applies to all similar cases.

There are many number systems. All of them have to do with a specified set of numbers and operations defined on that set. Because of the relative simplicity of natural numbers, we will continue to work, for a while, with this set. When BOOKLET 7 has been completed, all the properties of a typical number system will have been explained.

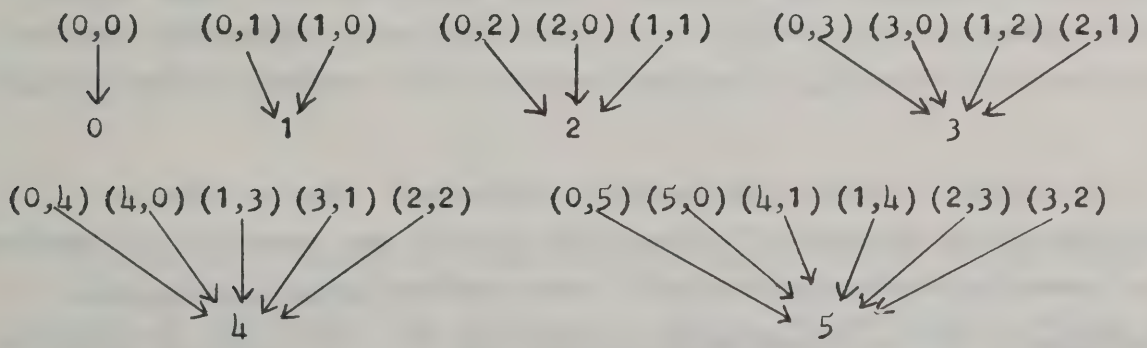
An operation on a set is a rule that assigns to each member (or each group of members) in that set a uniquely defined member in the same set. "Finding the successor of a member," for example, is an operation useful in the ordering of a set. If we are ordering the set N, we know that if a is a member of N, a + 1 is a description of the successor of a for every permissible replacement of a. "Finding the successor" is an operation on a single member of a set.

The majority of operations in arithmetic are binary operations. A binary operation on a set is a rule that assigns to each pair of members of the set a uniquely defined member in the same set. Addition is one such operation.

Binary operations can best be defined and understood in terms of ordered pairs. If this remark seems strange to you, reflect for a bit about the

frequent occurrence of ordered pairs in everyday experience. There are many determinants of order. One of them is time. Perhaps on Thursday you deposited x dollars in your bank account, and on the following Tuesday you decided to make a deposit of y dollars. There is an ordered pair, (x,y), that uniquely determines a number $x + y$ by means of a particular binary operation called addition for all permissible replacements of the variables. If x and y are natural numbers, $x + y$ is also a natural number. If we replace (x,y) by $(25,48)$ then $x + y$ becomes $25 + 48$, or 73. We say that $x + y$ is uniquely determined for each x and y. We can also say that the same binary operation maps $(48,25)$ onto 73. Addition, then, is a special kind of mapping.

You will remember, from Lesson 1, that if any two sets A and B are given, and if we decide to associate in any way whatsoever each member of B with a particular member of A, and if all the members of both sets are used, we have a mapping of A onto B. Whenever more than one member of a given set is mapped onto a single member of a second set, we refer to the mapping as many-to-one. Addition is really a many-to-one mapping of the Cartesian set (all possible ordered pairs) of natural numbers onto the set of natural numbers. The diagram below shows the ordered pairs mapped under addition onto the first 6 members of N.



The students may be interested in continuing this mapping. This mapping will provide new insight concerning the basic facts of addition.

Since addition is an operation that maps each pair of elements in N onto an element that is also in N, we say that the set of natural numbers is closed under addition. The mapping includes, of course, ordered pairs whose components are the same number. For example, $(3,3)$ is mapped by addition onto 6.

Some of the subsets of N are not closed under addition. The most familiar of these is $\{1, 3, 5, \cdots\}$. Since addition maps (1,3) onto 4, and 4 is not an odd number, we know that $\{1,3,5, \cdots\}$ is not closed under addition. $\{1,3,5, \cdots\}$ can be closed under the operation of "finding the successor," if we define the successor of an odd number a as a + 2 for each permissible replacement of a. Notice that this operation is not binary. The definition of the operation of subtraction involves the operation of addition. We shall find that N is not closed under subtraction. We begin by selecting an ordered pair of natural numbers such as (7,2), and we define the natural number that satisfies $7 = x + 2$ as the difference of 7 and 2. The ordered pair (7,2) is mapped onto 5, or 7 - 2, by this new operation because $7 = 5 + 2$. The condition $8 = x + 11$ cannot be satisfied by a member of N because the ordered pair (8,11) cannot be mapped under subtraction onto a member of N. Only those ordered pairs in $N \times N$ whose first component is greater than or equal to the second component can be mapped by subtraction onto a member of N. The number of ordered pairs mapped under subtraction onto any given member of N is infinite. Each member of $\{(0,0), (1,1), (2,2), \cdots\}$ is mapped onto 0. Each member of $\{(1,0), (2,1), (3,2), \cdots\}$ is mapped onto 1. Each member of $\{(2,0), (3,1), (4,2), \cdots\}$ is mapped onto 2. And so on. But such pairs as (0,1), (1,2), (2,3), (0,2), (1,3), and (2,4) are not included in the mapping. Subtraction then is a **many-to-one** mapping of a subset of $N \times N$ onto N. N is not closed under subtraction because this operation does not assign to <u>any</u> two members of N a uniquely defined element in N.

The ideas in Lesson 4 are basically important. Some of them are completely new to the students. To extend the content over several class sessions, break the lesson into several subtopics. These could be <u>Disjoint Sets and Union of Sets</u>, pp. 15-17, <u>The Operation of Addition</u>, pp. 17-20, and <u>The Operation of Subtraction</u>, pp. 20-22. Selections from the "On Your Own" exercises appropriate for each of these topics could be made. Another possibility is to use the developmental exercises exclusively until all the major ideas have been covered, then assign the "On Your Own" section as a final project.

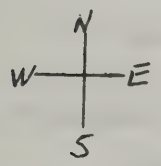
Here are a few supplementary problems that you may wish to use. They have to do with the union of sets.

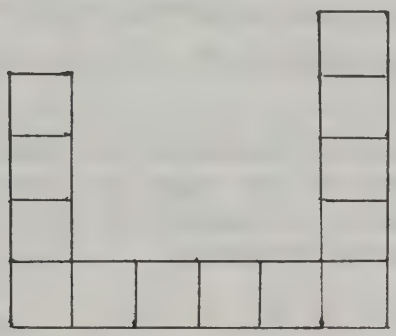
1 Steve attends English class on Monday, Wednesday, and Thursday of each week, and mathematics class on Tuesday, Thursday, and Saturday. Tabulate the set of days on which Steve attends either an English or a mathematics class. This set is the union of what two sets? Are they disjoint sets?

2 Jack attends English class on Monday, Wednesday, and Thursday of each week, and mathematics class on Tuesday, Friday, and Saturday. Tabulate the set of days on which Jack attends either an English or a mathematics class. This set is the union of what two sets? Are they disjoint sets?

3 The bedrooms in Mary's house are painted yellow, lavender, green, and pink. The bedrooms in Anne's house are blue and white. Are these two sets of colors disjoint? What number is associated with all the different colors used to paint the bedrooms in these two houses?

4 At Lakeview School a set of 6 classrooms faces south, a set of 4 classrooms faces west, and a set of 5 classrooms faces east. For the most economical arrangement should these three sets be disjoint? Make a diagram to illustrate your answer. Use your diagram to determine the number of classrooms associated with the union of the three sets. (For the most economical arrangement, the three sets of classrooms should not be disjoint. See the diagram below.)





5 A rectangular garden is to have a brick border with 50 bricks along each side and 30 bricks along each end. John said that 160 bricks would be needed for the project. But Harry, who had studied about sets that are not disjoint, claimed that 156 bricks would be sufficient. Whose answer is the correct one? Make a diagram to illustrate your conclusion.

6 While Joan was vacationing in California, she took a picture of a flamingo, a penguin, and a sea gull. At home in New England during the spring she had taken pictures of an oriole, a bluejay, a thrush, a woodpecker, a robin, and a sparrow. Her friend, Sue, had a collection of pictures, too. Her pictures were of a duck, a robin, a woodpecker,

a sparrow, and an ostrich. Tabulate the set of all the different birds in these collections. This set is the union of what three sets?

Page 15

Answers for Exercise C — Set T is made up of the elements that are members of either set R or set S. Note that any element that is a member of both set R and set S is named only once in the tabulation of set T.

Exercise D — Since Hong Kong is a member of L, it will be a member of the union of K and L.
Since Rome is a member of neither K nor L, it will not be a member of the union of K and L.

Exercise E — The set tabulated is the union of K and L because it is the set of all members of K together with all members of L; it does not include any other members.

Exercise F — The set tabulated does not include all the members of set K.

Exercise H — The sentence expresses a true statement because {Berlin, Paris, London, Hong Kong, Tokyo} is the correct tabulation of the union of K and L.

Exercise I — Tabulation of A: {0, 1, 2}
Tabulation of B: {12, 13, 14}

Page 16

Answers for Exercise J — Since 11 is a member of neither A nor B, it is not a member of the union of A and B.
Since 2 is a member of set A, it is a member of the union of A and B.

Exercise K — {0, 1, 2, 12, 13, 14}
$A \cup B$ is a name for the union of A and B, and this is the set tabulated. So we can say $A \cup B = \{0, 1, 2, 12, 13, 14\}$.

Exercise L — $F = D \cup E$ is a true statement because F and $D \cup E$ are different names for the same set.

Exercise M — {2, 4, 8, 16}
Note: $A \cup A = A$.

Exercise N — No; Sally is a member of both R and S.
Yes; no member of K is also a member of L.

Page 16 (cont.)

Answers for Exercise O — No

Set A and set B are tabulated in Exercise I above. The tabulations show that they have no common members.

Exercise P — None; any two of the sets in Display 1 have at least one common member.

Yes, because K and L have no common members

None; any two of the sets in Display 5 have at least one common member.

Exercise Q — Since {0, 1} and {2, 4} have no common member, they are disjoint sets.

{0, 1, 2, 4} The number 4

Exercise R — No, because 0 is a member of both sets

{0, 1, 2} The number 3

Page 17

Answers for Exercise A — Since set A and set B have no common members, they are disjoint.

The number 2 is associated with set A because this set has two members.

The number 3 is associated with set B because this set has three members.

Exercise B — The number of members in set A is two, and the number of members in {1, 3} is two. So $\underline{n}(A)$ and $\underline{n}\{1,3\}$ are, in this instance, names for the number 2.

Exercise D — {1, 3, 0, 2, 4}

Exercise E — A and B

Exercise F — Yes

The tabulation in Exercise D above shows that $\underline{n}(A \cup B)$ is 5.

Exercise G — Since E and F have no common members, they are disjoint sets.

Exercise H — {Kentucky, Florida, Georgia, Alabama, Louisiana, Texas}

Exercise I — A: One is the number of members in E.

B: Five is the number of members in F.

C: The sum of 1 and 5 is the number of members in the union of E and F.

Page 17

Answers for Exercise I (cont.)

Reference to Display 9 will show that the sentences labeled A and B express true statements.

The tabulation in Exercise H above shows that the sentence labeled C expresses a true statement.

The sum of 1 and 5 is the number associated with the union of E and F. This union, in this example, is {Kentucky, Florida, Georgia, Alabama, Louisiana, Texas}.

Exercise J

Yes, since $\underline{2+3} = \underline{n}(A \cup B)$

The sum of the number associated with E and the number associated with F is equal to the number associated with the union of E and F.

Page 18

Answers for Exercise K

$S \cup T$

Exercise L

$S \cup T$

Exercise M

$S \cup T$

Exercise N

There are many correct answers. The following are typical.

G = {Superior, Erie, Ontario, Michigan, Huron}.

H = {2, 4, 6, 8, 10, 12, 14, 16, 18}.

$G \cup H$ = {Superior, Erie, Ontario, Michigan, Huron, 2, 4, 6, 8, 10, 12, 14, 16, 18}.

Exercise O

$G \cup H$

Since 5 + 9 is a name for 14, and $\underline{n}(G \cup H)$ is, in this example, a name for 14, $\underline{5 + 9} = \underline{n}(G \cup H)$ is a true statement.

Exercise P

No, because A and B are not disjoint sets

They have two members, namely 18 and 19, in common.

Page 19

Answers for Exercise A

The number of questions that Jack answered correctly on Monday's history test

The number of questions that Jack answered correctly on Friday's history test

Exercise C

(7,9) (9,10) (10,9)

Exercise G

Yes

The mapping is one-to-one because each member of the set of ordered pairs is mapped onto exactly one member of the set of students, and every member of the set of students is used in the mapping.

Page 19 (cont.)
Answers for Exercise H (9,8) The number 17

Exercise I Yes
Each ordered pair was mapped onto the number that is the sum of its components.

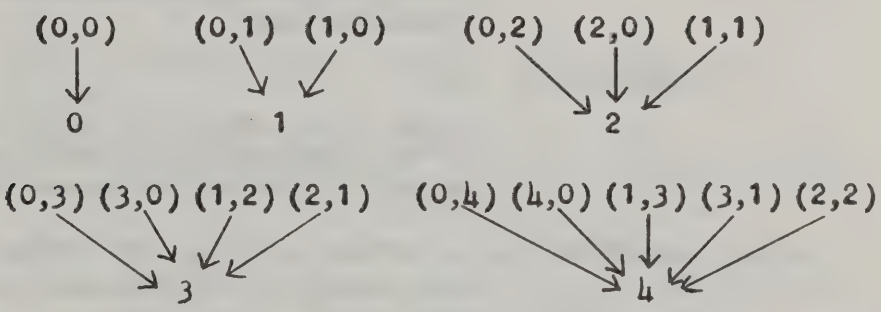
Exercise J Each of the ordered pairs given should be mapped onto the number seven, because the sum of the first and second components, for each of the pairs, is seven.

Page 20
Answers for Exercise L The sum of their components, in each case, is eight.
(0,8), (8,0), (2,6), (3,5), (5,3), (4,4)

Exercise M The sum of their components, in each case, is four.
(0,4), (3,1), (2,2)

Exercise N The mappings in Display 14 and Display 15 are many-to-one since more than one ordered pair is mapped onto the same sum. For example, (9,10) and (10,9) are each mapped onto 19. (7,1) and (6,2) are each mapped onto 8.

Exercise O The natural numbers
The mapping begins thus:



Exercise P The number used as the replacement for c.

Exercise A Yes; 3 + 10 = 13 is a true statement.

Exercise B 10
Yes; 10 + 3 = 13 is a true statement.

Exercise C 5 6

Exercise D In each case, the sum of the number onto which the ordered pair is mapped and the second component of the pair is equal to the first component.
5 + 6 = 11. 6 + 5 = 11.

Page 20 (cont.)

Answers for Exercise E — Finding this number amounts to finding what number can be added to the second component of (8,5) to get the first component. $3 + 5 = 8$ indicates that this number is 3. So (8,5) is mapped onto 3.

Exercise F — $6 + 2 = 8$ indicates that the number 6 can be added to the number 2 to get the sum 8. So (8,2) is mapped onto 6.

Exercise G — Yes, because 6 is the result of subtracting 2 from 8

Page 21

Answers for Exercise H — 2

7; because the sum of 7 and 2 is 9

Exercise J

(10,2)	(10,8)	(14,3)	(20,20)
↓	↓	↓	↓
8	2	11	0

Exercise K — The number used as the replacement for c.

Exercise L — Yes, in this case we are using only natural numbers. We begin with two natural numbers, namely the components of an ordered pair. Any operation on two numbers is a binary operation, so subtraction is included in this classification.

Exercise M — Yes

Think of the members of {(6,0), (7,1), (8,2), (9,3)}. If the operation being performed is subtraction, each member of this set is mapped onto the number 6.

Exercise N — 4, because $5 - 1 = 4$ is a true statement

Note that $5 = 4 + 1$ is also a true statement.

Exercise O

(5,1)	(1,5)	(9,3)	(3,9)
↓	↓	↓	↓
6	6	12	12

Exercise P — Yes; there is just one natural number for any given ordered pair.

For example, the ordered pair (7,2) is mapped onto the number 9 by addition. It cannot be mapped by addition onto any number other than 9.

Exercise Q — Yes

Page 21

Answers for Exercise Q (cont) Since $5 = 4 + 1$ is a true statement, the number onto which $(5,1)$ is mapped under subtraction is 4.

Exercise R $5 - 1$, or 4, is the difference of 5 and 1 because $5 = 4 + 1$ is a true statement.

Exercise S The number 1

Exercise T There is no natural number such that the sum of that number and 5 is 1.

Exercise U Yes

$(5,2)$ is mapped onto 3 by subtraction because $5 = 3 + 2$ is a true statement.

Exercise V No, because there is no natural number such that the sum of that number and 5 is 2

Exercise W There are many correct answers. The following are typical of them.

$$\begin{array}{ccc} (7,2) & (55,28) & (1,0) \\ \downarrow & \downarrow & \downarrow \\ 5 & 27 & 1 \end{array}$$

Exercise X There are many correct answers. The following are typical.

$(2,9)$ This pair is not mapped onto a natural number by subtraction since there is no natural number to which 9 can be added to get a sum of 2.

$(6,21)$ This pair is not mapped onto a natural number by subtraction since there is no natural number to which 21 can be added to get a sum of 6.

$(0,8)$ This pair is not mapped onto a natural number by subtraction since there is no natural number to which 8 can be added to get a sum of 0.

Page 22

Answers for Exercise Y No, not when the first component is less than the second component.

Exercise Z The difference $\underline{a} - \underline{b}$ will not be a natural number only if the replacement for $\underline{a}$ is less than the replacement for $\underline{b}$.

Page 22 (cont.)

Answers for Exercise A

The sets can be described correctly in more than one way.

A is the set of counting numbers from 1 through 6.

or

$A = \{x \mid x > 0 \wedge x < 7\}$. $U = N$.

B is the set of odd natural numbers.

or

$B = \{x \mid x \text{ is an odd number}\}$. $U = N$.

Exercise B — Set A and set B are subsets of N because each of their members is also a member of N.

Exercise C — $2+3$, or 5, is a member of A, but $3+4$, or 7, is not a member of A.

Exercise D — No

Exercise C above contains one exception. Another is $5+6$.

Exercise E — No

The sum of any two members of A is not necessarily a member of A.

Exercise F — Since $3-3 = 0$ is a true statement, $3-3$ is not a member of A.

Exercise G — Set A is not closed under subtraction.

The difference of two members of A is not necessarily a member of A. For example, $3-5$ is not a member of A.

Exercise H — 3, 5, and 9 are members of B because B includes all odd natural numbers.

$3+5$ is not in B because 8 is not an odd number. For the same reason, $5+9$ is not in B.

Exercise I — B is not closed under addition. Exercise H shows that the sum of two members of B is not necessarily a member of B.

Exercise J — B is not closed under subtraction, since the difference of two members of B is not necessarily a member of B. For example, $5-7$ is not a member of B.

Exercise K — C is closed under addition because the sum of any two even natural numbers is also an even natural number.

Page 22

Answers for Exercise K (cont.) C is not closed under subtraction because the difference of two even natural numbers is not necessarily an even natural number. For example, $4-6$ is not an even natural number.

Exercise L The universe for a and b is the set of even natural numbers. For each replacement of a and b, there is a number $a+b$ that is a member of the set of even natural numbers.

Pages 22-23

Answers for "On Your Own" exercises

1 $A \cup B = \{1, 2, 3, 5\}$.

2 $A \cup B = \{0, 1, 2, 3, 4, 5, 6, 8\}$.

3 $A \cup B = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$.

4 $A \cup B = \{6, 8, 12, 16, 18, 24, 30, 32, 36, 40, \cdots\}$.

5 $\{1, 2, 3, 4, 6, 8, 10\}$

6 $\{2, 4, 6, 8, 10\}$

7 $\{1, 2, 3, 4, 6, 8, 10\}$

8 $\{2, 4, 6, 8, 10, 12, 16, 20, 24, \cdots\}$.

9 $\{2, 4, 6, 8, 10, 15, 20\}$

10 $\{2, 4, 6, 8, 10, 450\}$

11 $\{2, 4, 6, 8, 10\}$ and $\{1, 2, 3\}$ are not disjoint sets.
$\{2, 4, 6, 8, 10\}$ and $\{2\}$ are not disjoint sets.
$\{2, 4, 6, 8, 10\}$ and $\{1, 3\}$ are disjoint sets.
$\{2, 4, 6, 8, 10\}$ and $\{4, 8, 12, 16, \cdots\}$ are not disjoint sets.
$\{2, 4, 6, 8, 10\}$ and $\{10, 15, 20\}$ are not disjoint sets.
$\{2. 4, 6, 8, 10\}$ and $\{450\}$ are disjoint sets.

12 Whether or not the particular set C and set D are disjoint sets

13 $28 = \underline{n}(E \cup F)$.

14 $28 = \underline{n}(F \cup E)$.

15 $27 = \underline{n}(G \cup E)$.

16 $20 = \underline{n}(F \cup F)$.

Answers for "On Your Own" exercises (cont.)

17 $78 = \underline{n}(E \cup H)$.

18 $47 = \underline{n}[(G \cup E) \cup F]$.

19 $98 = \underline{n}[E \cup (F \cup H)]$.

20 $117 = \underline{n}[(E \cup F) \cup (G \cup H)]$.

21

(70,50)	(29,54)	(x,x)	(x,2x)	(x,4x)
↓	↓	↓	↓	↓
120	83	2x	3x	5x

Examples:

If the number 5 replaces $\underline{x}$, the ordered pair (x,x) becomes (5,5), and $2\underline{x}$ becomes 10. Addition maps the pair (5,5) onto 2(5), or 10.

If the number 6 replaces $\underline{x}$, the ordered pair (x,2x) becomes (6,12), and $3\underline{x}$ becomes 18. Addition maps the pair (6,12) onto 3(6), or 18.

If the number 9 replaces $\underline{x}$, the ordered pair (x,4x) becomes (9,36), and $5\underline{x}$ becomes 45. Addition maps the pair (9,36) onto 5(9), or 45.

22

(50,25)	(78,40)	(x,x)	(2x,x)	(4x,x)
↓	↓	↓	↓	↓
25	38	0	x	3x

Examples:

If the number 7 replaces $\underline{x}$, the ordered pair (x,x) becomes (7,7). Subtraction maps the pair (7,7) onto 7 - 7, or 0.

If the number 5 replaces $\underline{x}$, the ordered pair (2x,x) becomes (10,5). Subtraction maps the pair (10,5) onto 5.

If the number 11 replaces $\underline{x}$, the ordered pair (4x,x) becomes (44,11), and $3\underline{x}$ becomes 33. Subtraction maps the pair (44,11) onto 3(11), or 33.

23 18, 21, 24, 27, 30

24 {3, 6, 9, 12, 15, ···} is closed under addition. This is the set of natural numbers greater than 0 and that have 3 as a factor. The sum of any two members of the set also has 3 as a factor.

The set is not closed under subtraction, because the difference of any two of its members is not necessarily a natural number.

25 The set of odd numbers is not closed under either addition or subtraction.

26 The set of even numbers less than 14 is not closed either under addition or subtraction.

27 The set of all even numbers is closed under addition but not closed under subtraction.

Answers for "On Your Own" exercises (cont.)

28 The set of natural numbers greater than 50 is closed under addition but not closed under subtraction.

29 The set of natural numbers less than 50 is not closed under either addition or subtraction.

30 The set of odd numbers greater than 99 is not closed under either addition or subtraction.

LESSON 5

Pages 23-28 Notes

The binary operations--addition, subtraction, multiplication, and division--are referred to frequently as the fundamental operations of arithmetic. In describing the properties of a number system, only two of these operations, addition and multiplication, are used. Subtraction and division can be defined in terms of addition and multiplication.

From their work in Lesson 4, the students are familiar with addition as a binary operation on N, and with subtraction as a binary operation on a subset of N. In this lesson they will see how multiplication and division differ from addition and subtraction, and how multiplication and division are similar to addition and subtraction.

Addition is defined by means of the union of two disjoint sets. Multiplication is defined by means of the Cartesian set of two given sets that may or may not be disjoint. The number onto which two numbers are mapped by addition is called their sum. The number onto which two numbers are mapped by multiplication is called their product. Both addition and multiplication are many-to-one mappings of all the members of $N \times N$ onto N.

Consider the sets $A = \{a, b, c\}$ and $B = \{d, e\}$. Sets A and B are disjoint sets. The Cartesian set, $A \times B$, is $\{(a,d), (a,e), (b,d), (b,e), (c,d), (c,e)\}$. Since $3 = \underline{n}(A)$ and $2 = \underline{n}(B)$, 3×2, or $6 = \underline{n}(A \times B)$. Now consider the sets $A = \{a, b, c\}$ and $C = \{b, c\}$. Sets A and C are not disjoint sets. The Cartesian set, $A \times C$, has 6 members.

Multiplication, like addition, maps each ordered pair in N × N onto only one member of N. The components of the ordered pairs mapped onto a particular number by multiplication are factors of that number. For example, (1,12), (12,1), (2,6), (6,2), (3,4), (4,3) are each mapped by multiplication onto the number 12. 1,2,3,4,6, and 12 are factors of 12. (1,13), (13,1) are the members of N × N that are mapped by the same operation onto the number 13. 1 and 13 are factors of 13. 13 is a prime number. Only two pairs are mapped onto a prime number because a prime number has only two factors. If a is replaced by a prime number, the ordered pairs mapped onto a are (a,1) and (1,a).

An excellent exercise for increasing understanding of factoring is provided by the mapping of ordered pairs onto members of N. The students could be asked to find all the ordered pairs mapped by multiplication onto the number 1000 or any other natural number.

Now we shall define division in terms of multiplication. We shall also find that N is not closed under division. We will select an ordered pair of natural numbers, such as (10,5). We define the natural number that satisfies the condition $10 = x \times 5$ as the quotient of 10 and 5. (10,5) is mapped onto 2, or $10 \div 5$, by this new operation because $10 = 2 \times 5$ is a true statement. The condition $6 = x \times 12$ cannot be satisfied by a member of N; so (6,12) cannot be mapped by division onto a member of N. The condition $18 = x \times 7$ cannot be satisfied by a member of N; so (18,7) cannot be mapped by division onto a member of N. The condition $6 = x \times 0$ cannot be satisfied by a member of N; so (6,0) cannot be mapped by division onto a member of N.

To identify members of N × N that can be mapped by division onto N, we must decide if the second component is a factor of the first component. If a and b in $a \div b$ are replaced by members of N (b cannot be replaced by 0), and if the replacement for b is a factor of the replacement for a, then after replacement (a,b) will be mapped by division onto a member of N. Remember that every member of N is a factor of 0.

In Lesson 12 there is an explanation of why the number 0 can never be used as a divisor. This is a restriction of fundamental importance in mathematics,

though the necessity for making it was not realized until about the middle of the nineteenth century.

In Lesson 4 it was observed that the number of ordered pairs mapped by subtraction onto any given member of N is infinite. This is also true of the many-to-one mapping determined by division. For example, each member of $\{(0,1), (0,2), (0,3), \cdots\}$ is mapped onto 0; each member of $\{(1,1), (2,2), (3,3), \cdots\}$ is mapped onto 1; each member of $\{(2,1), (4,2), (6,3), \cdots\}$ is mapped onto 2, and so on.

To review and to increase understanding of the division basic facts, use the mapping of pairs by the operation of division onto the members of N. Some of the students will discover that the number of ordered pairs mapped onto a given member of N has no limit.

Page 23
Answer for Exercise A Yes The diagram below shows why.

Page 24
Answers for Exercise B Yes
The first sentence expresses the fact that the number 2 is equal to the number of members in set A, and the second expresses the fact that the number 3 is equal to the number of members in set B.

Exercise D Yes
This can be confirmed by reference to Display 1.

Exercise E $D \times E = \{(0,0), (0,1), (0,2), (0,3), (0,4), (0,5), (1,0), (1,1), (1,2), (1,3), (1,4), (1,5), (2,0), (2,1), (2,2), (2,3), (2,4), (2,5), (3,0), (3,1), (3,2), (3,3), (3,4), (3,5)\}$.

Exercise F 4 6 4×6

Exercise K S×T

Exercise L S×T

Page 24 (cont.)
Answer for **Exercise M** S×T, for every permissible replacement of s and t

Page 25
Answers for Exercise N Any two sets with 4 and 2 members respectively will do.
The cross product will always have 8 members. To construct it, use the type of pattern in Exercise A above.

Exercise O A×B
4 and 2

Exercise P 0 4

Exercise Q The empty set
The empty set has no members to be paired with the members of {2, 3, 4, 5}. So there are no members of the Cartesian set. Therefore, this Cartesian set is the empty set.

Exercise R Yes
This follows from the association of the number 0 with the empty set, and the fact that the cross product of the empty set and any other set has no members.

Exercise S No, because whether or not two sets are disjoint does not affect the number of ordered pairs in their cross product

Exercise A Yes
Each ordered pair of natural numbers is mapped onto the natural number that is the product of its components.

Exercise C 16 16 16
The mapping of several different ordered pairs onto the same member of N makes the mapping many-to-one instead of one-to-one.
(2,8) (16,1)

Exercise D 15 80

Exercise E Multiplication is a binary operation on set N because it maps each ordered pair in N × N onto only one member of N.

Exercise F Yes; 2 × 3 = 6.

Exercise G 3 3 × 2 = 6. The first component

Page 26

Answers to Exercise H

In each case the product of the number onto which the pair is mapped and the second component of the pair is equal to the first component.

A: $5 \times 2 = 10$. B: $5 \times 5 = 25$.

Exercise I

The statement $3 \times 7 = 21$ tells us that 3 times the second component of the ordered pair (21,7) equals the first component of the ordered pair. So we know that (21,7) can be mapped under division onto 3.

Exercise J

5 3

Exercise K

Yes

The statement $2 \times 3 = 6$ indicates that 2 is the quotient onto which an ordered pair whose first component is 6 and whose second component is 3 has been mapped.

Explanation for the other three examples in Display 6 is similar to the one given.

Exercise L

4 2 1 5

Exercise M

The ordered pair (10,5) is mapped onto 2.

The statement $2 \times 5 = 10$ indicates that an ordered pair whose first component is 10 and whose second component is 5 has been mapped onto their quotient 2.

Exercise N

The replacement for $\underline{c}$

Exercise O

3 11

Exercise P

Division is a binary operation on a proper subset of N. A binary operation on a set is a rule that assigns to each pair of elements of that set a uniquely defined element in the same set. The operation of division does this for some, but not all, the ordered pairs that are members of $N \times N$.

Exercise Q

Division is a many-to-one mapping of a proper subset of $N \times N$ onto N.

Each member of $\{(0,1), (0,2), (0,3), \cdots\}$ is mapped by division onto 0.

Each member of $\{(1,1), (2,2), (3,3), \cdots\}$ is mapped by division onto 1.

Each member of $\{(2,1), (4,2), (6,3), \cdots\}$ is by division mapped onto 2.

This mapping can be continued indefinitely.

Page 26 (cont.)

Answers for Exercise A — 5, because $10 = 5 \times 2$ is a true statement

Exercise B — 12

Exercise C — We cannot use division to map (3,4) onto a natural number because there is no member of N that satisfies the condition $3 = x \times 4$.

Exercise D — Yes. The number is 44 because $11 \times 4 = 44$ is a true statement.

No, because there is no member of N that satisfies the condition $11 = x \times 4$

Exercise E — Yes; multiplication maps a member of $N \times N$ onto N.

No; division maps a member of a proper subset of $N \times N$ onto N.

Page 27

Answers for Exercise F — The quotient a ÷ b is a natural number when the replacement for b is a factor of the replacement for a, and the replacement for b is not 0.

Exercise G — A is the set of even natural numbers that are less than 6.

B is the set of even natural numbers.

Exercise H — Since $0 \times 2 = 0$ is a true statement and 0 is a member of A, 0×2 is also a member of A.

2×4 is not a member of A because 8 is not a member of A.

Exercise I — No, because there are members of $A \times A$, such as (2,4), that do not map by multiplication onto members of A

Exercise J — Since $4 \div 2 = 2$ is a true statement, 4÷2 is a member of A.

Since there is no member of A that satisfies the condition $2 = x \times 4$, 2÷4 is not a member of A.

Exercise K — No, because some members of $A \times A$, such as (2,4), do not map by division onto members of A

Exercise L — Since 6, 12, and 20 are even natural numbers, they are members of B.

Multiplication maps 6×12 onto 72, and 72 is an even number. So 6×12 is a member of B.

Multiplication maps 20×6 onto 120, which is an even number. So 20×6 is a member of B.

Page 27
Answers for Exercise L (cont.) Since multiplication maps each member of B × B onto a member of B, the set is closed under multiplication.

Exercise M B is not closed under division because there are members of B × B, such as (2,4), that do not map by division onto members of B.

Exercise N Yes, because multiplication maps each member of C × C onto a member of C

No, because there are some members of C × C, such as (1,3), that division does not map onto members of C

Exercise O The universe for a and b is the set of odd natural numbers. For each replacement of a and b there is a number axb that is a member of the set of odd natural numbers.

Pages 27-28
Answers for "On Your Own" exercises

1 $A \times B = \{(0,5), (0,7), (0,9), (1,5), (1,7), (1,9)\}$.

2 $B \times A = \{(5,0), (5,1), (7,0), (7,1), (9,0), (9,1)\}$.

3 $A \times A = \{(0,0), (0,1), (1,0), (1,1)\}$.

4 $B \times C = \{(5,10), (7,10), (9,10)\}$.

5 $C \times D = \{\ \}$.

6 $A \times C = \{(0,10), (1,10)\}$.

7 $D \times A = \{\ \}$.

8 $B \times D = \{\ \}$.

9 $D \times D = \{\ \}$.

10 A: 2 members C: one member
B: 3 members D: no members

11 $A \times B$: 6 members $A \times C$: 2 members
$B \times A$: 6 members $D \times A$: no members
$A \times A$: 4 members $B \times D$: no members
$B \times C$: 3 members $D \times D$: no members
$C \times D$: no members

Answers for "On Your Own" exercises (cont.)

12	30	15	600	18	0
13	30	16	10,000	19	0
14	25	17	500	20	36
21	2450	23	60	25	$1\underline{x}$
22	150	24	$3\underline{x}$	26	0

27 Examples
Note: The number 5 is selected as a replacement for $\underline{x}$. Any other natural number could also be used.

In Exercise 24, if 5 replaces $\underline{x}$, the ordered pair becomes (5,3). Multiplication maps (5,3) onto 15. $3(5)$ also equals 15.

In Exercise 25, if 5 replaces $\underline{x}$, the ordered pair becomes (1,5). Multiplication maps (1,5) onto 5. $1(5)$ also equals 5.

In Exercise 26, if 5 replaces $\underline{x}$, the ordered pair becomes (5,0). Multiplication maps (5,0) onto 0. $5(0)$ also equals 0.

28	2	30	10	32	1
29	5	31	1	33	$\underline{x}$

34 Examples
Note: Any member of the universe N could be used to replace $\underline{x}$. 7 is chosen for these examples.

In Exercise 32, if 7 replaces $\underline{x}$, the ordered pair becomes (7,7). Since $7 = y \times 7$ is satisfied by the number 1, division maps (7,7) onto 1.

In Exercise 33, if 7 replaces $\underline{x}$, the ordered pair becomes (7,1). Since $7 = y \times 1$ is satisfied by the number 7, division maps (7,1) onto 7.

35 The set of odd numbers is closed under multiplication because the product of any two odd numbers is a member of the same set.

36 The set of even numbers is closed under multiplication because the product of any two even numbers is a member of the same set.

37 The set of natural numbers greater than 20 is closed under multiplication because the product of any two members of the set is a member of the same set.

38 The set of natural numbers less than 20 is not closed under multiplication because the product of any two members of the set is not necessarily a member of the same set. For example, $\underline{3 \times 11}$ is not less than 20.

39 The set of natural numbers from 51 through 60 is not closed under multiplication because the product of any two members of the set is not a member of the same set. For example, 51×60 is not a member of the set.

40 None of the sets described in Exercises 35 through 39 is closed under division.

Division does not map (7,3) onto the set of odd numbers.

Division does not map (8,8) onto the set of even numbers.

Division does not map (50,25) onto the set of natural numbers greater than 20.

Division does not map (18,7) onto the set of natural numbers less than 20.

Division does not map (54,55) onto the set of natural numbers from 51 through 60.

LESSON 6

Pages 28-34 Notes

Since the operation of addition involves the union of disjoint sets, the properties of addition may be derived from such unions. The first property that we shall consider is commutativity. It is expressed in words at the end of P. 29. To establish commutativity as a property of the addition of natural numbers, we note that if A and B are any disjoint sets, $A \cup B = B \cup A$. This means that the union of A and B is the same set as the union of B and A. Consequently, the number associated with $A \cup B$ is the same as the number associated with $B \cup A$. We know also that if A and B are disjoint sets and $\underline{a} = \underline{n}(A)$ and $\underline{b} = \underline{n}(B)$, then $\underline{a} + \underline{b} = \underline{n}(A \cup B)$. Also $\underline{b} + \underline{a} = \underline{n}(B \cup A)$. Therefore, $\underline{a} + \underline{b} = \underline{b} + \underline{a}$ for every replacement of $\underline{a}$ and $\underline{b}$ from N.

To increase interest in the commutative property of addition, have the students construct an addition table similar to the one at the top of the next page. Give the students a chance to discover the symmetry of the arrangement of the numerals with respect to a diagonal line drawn through it. This is true of all tables for a commutative operation.

+	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	6
2	2	3	4	5	6	7
3	3	4	5	6	7	8
4	4	5	6	7	8	9
5	5	6	7	8	9	10

The definition of multiplication involves the number of members in the Cartesian set of two given sets. But if A and B are sets, $A \times B = B \times A$ is not a true statement because $A \times B$ is not the same set as $B \times A$. These two sets are equivalent, however, because they can be put into one-to-one correspondence. This correspondence can be set up so that a pair such as (3,5) is linked to the pair (5,3). Display 12, page 31, illustrates such a one-to-one correspondence. Since $n(A \times B) = n(B \times A)$ is a true statement, we know that $a \times b$ will be the same number as $b \times a$ after replacement if $a = n(A)$ and $b = n(B)$ are true after replacement. The commutative property of multiplication of natural numbers is stated on page 32.

A supplementary activity is the construction of a multiplication table to observe symmetry. Such a table is shown below.

×	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	2	3	4	5
2	0	2	4	6	8	10
3	0	3	6	9	12	15
4	0	4	8	12	16	20
5	0	5	10	15	20	25

A binary operation on a set is simply a rule that assigns to any two members of a set a single member of the same set. We can invent such operations and examine them to see what properties they have. "On Your Own" Exercises 21-32, page 34, illustrate this point. In Exercise 21, the symbol * defines an operation that maps a * b onto a + 2b when a and b are replaced by natural numbers. If a is replaced by 5 and b is replaced by 8, the rule indicated that 5 * 8 = 5 + 2(8) is a true statement. Similarly, if a is replaced by 8 and b is replaced by 5, 8 * 5 = 8 + 2(5) is a true statement.

A table for this operation is shown below. Notice the absence of symmetry of arrangement of numerals with respect to the diagonal. This shows that the operation "*" does not have the commutative property. The presence of a numeral in every space indicates that N is closed under the operation "*."

*	0	1	2	3	4	5
0	0	2	4	6	8	10
1	1	3	5	7	9	11
2	2	4	6	8	10	12
3	3	5	7	9	11	13
4	4	6	8	10	12	14
5	5	7	9	11	13	15

For instructional purposes this lesson can be divided into two or more parts. The first class session could be spent on the Commutative property of Addition, pp. 28-30. Follow this with the Commutative Property of Multiplication, pp. 30-33. Whether or not subtraction is a commutative operation could be discussed in connection with the first of these topics since subtraction is defined in terms of addition. Comparable material for division might end the discussion on the second day. If you wish to spend a third day on the material in the lesson you could use the time for the construction of tables such as have been suggested, and for "On Your Own" Exercises 21-32.

Page 28

Answers for Exercise B — Sets M and N are disjoint because they have no common members.

Exercise C — 4 3

Exercise D — $M \cup N$ = {Dick, Tony, Bob, Paul, center, guard, quarterback}.

Exercise E — The sentence expresses a true statement because the sum of the numbers associated with two disjoint sets is the number associated with the union of these sets. Since M and N are disjoint and $4 = \underline{n}(M)$ and $3 = \underline{n}(N)$, then the sum $4 + 3$ is the number associated with $\underline{n}(M \cup N)$.

Exercise F — $N \cup M$ = {center, guard, quarterback, Dick, Tony, Bob, Paul}.

Exercise G — Since N and M are disjoint sets and $3 = \underline{n}(N)$ and $4 = \underline{n}(M)$, the sum $3 + 4$ is the number associated with $\underline{n}(N \cup M)$.

Exercise H — The two sets are the same because they have the same members. Changing the order in which the members are named does not change the set.

Page 29

Answers for Exercise I — Yes, because $M \cup N$ is the same set as $N \cup M$

Exercise J — Yes, since $M \cup N$ is the same set as $N \cup M$

Exercise K — Yes; they are associated with the same set.

Yes, because just one natural number is associated with a set

Exercise L — D is the set of odd natural numbers from 1 through 9.

E is the set of even natural numbers from 10 through 20.

These sets are disjoint because they have no common members.

Exercise M — 5 6

Exercise N — $D \cup E$ = {1, 3, 5, 7, 9, 10, 12, 14, 16, 18, 20}.

$E \cup D$ = {10, 12, 14, 16, 18, 20, 1, 3, 5, 7, 9}.

Since $E \cup D$ has the same members as $D \cup E$, it is the same set as $D \cup E$.

Exercise Q $B \cup A$ will be the same set as $A \cup B$ because the same elements are included in both of these unions.

Exercise R $A \cup B$ $B \cup A$

Yes, the set $A \cup B$ or $B \cup A$

Page 30

Answers for Exercise S In statement A, $\underline{a}$ is replaced by 6, and $\underline{b}$ is replaced by 9.
In statement B, $\underline{a}$ is replaced by 52, and $\underline{b}$ is replaced by 30
In statement C, $\underline{a}$ is replaced by 89, and $\underline{b}$ is replaced by 10

Each statement is true because the commutative property of addition for natural numbers states that for each replacement of $\underline{a}$ and $\underline{b}$ by a member of N, a true statement is obtained from $\underline{a} + \underline{b} = \underline{b} + \underline{a}$.

Exercise T Yes, the number 15

Exercise U Yes

$(\underline{a},\underline{b})$ is mapped onto $\underline{a + b}$, and $(\underline{b},\underline{a})$ is mapped onto $\underline{b + a}$. The commutative property of addition for natural numbers states that $\underline{a} + \underline{b} = \underline{b} + \underline{a}$ is a true statement, for each replacement of $\underline{a}$ and $\underline{b}$ from N.

Exercise V Yes
Suppose the number that replaces $\underline{a}$ and $\underline{b}$ is 4.
$4 + 4 = 4 + 4$ is a true statement.

Exercise B F is the set whose members are 13 and 14.

G is the set whose members are 0, 1, and 2.

Exercise C 2 3

Exercise D $F \times G = \{(13,0), (13,1), (13,2), (14,0), (14,1), (14,2)\}$.

Set F Set G

Exercise E Yes; $\underline{2 \times 3}$ is the number of members in $\underline{F \times G}$.

Exercise F $G \times F = \{(0,13), (0,14), (1,13), (1,14), (2,13), (2,14)\}$.

Set G Set F

Exercise G Yes; $\underline{3 \times 2}$ is the number of members in $\underline{G \times F}$.

Exercise H No, because the two sets are not made up of the same ordered pairs

Page 31

Answers for Exercise J (13,0) is mapped onto (0,13), and (0,13) is mapped onto (13,0).

Follow Display 12 to construct the appropriate response for the remaining five members of the sets.

Exercise K Each ordered pair in F×G is mapped onto the member of G×F that has the same numbers as components but in reverse order. The same rule is used in mapping the members of G×F onto the members of F×G.

Exercise L Since there is a one-to-one mapping of set F×G onto set G×F and there is a one-to-one mapping of set G×F onto set F×G, the two sets are in one-to-one correspondence.

The two sets are equivalent because any two sets that can be put into one-to-one correspondence are equivalent.

Exercise M Yes; any two equivalent sets have the same number of members.

Exercise N Yes, because F×G and G×F have the same number of members

Yes
The number of members in set F×G is 2×3, and the number of members in G×F is 3×2. These two numbers are the same.

Exercise A 3 4

Note: The answers to Exercises B through G are similar to those given for Exercise D through M above. The ideas are identical.

Exercise H Yes

Set B×A can always be constructed from set A×B by interchanging the components of the ordered pairs that are members of A×B. The two sets can always be put in one-to-one correspondence by the matching of (a,b) with (b,a) and of (b,a) with (a,b) for all permissible replacements of the variables.

Page 32

Answers for Exercise I a×b is associated with A×B. b×a is associated with B×A.

Since A×B and B×A are equivalent sets, a×b and b×a represent the same number.

The answer to each question in this exercise is yes.

Exercise J Otherwise $2\underline{m}$ would become 24 when $\underline{m}$ is replaced by 4.

Exercise K No, because $7 \cdot 8$ is a symbol for 56, which is a different number from 78.

Exercise L In statement A, $\underline{a}$ is replaced by 7, and $\underline{b}$ is replaced by 8.

In statement B, $\underline{a}$ is replaced by 29, and $\underline{b}$ is replaced by 16.

In statement C, $\underline{a}$ is replaced by 115, and $\underline{b}$ is replaced by 21.

Each statement is true because the commutative property of multiplication states that for each replacement of $\underline{a}$ and $\underline{b}$ a true statement is obtained from $\underline{ab} = \underline{ba}$.

Exercise M Yes; (7,8) is mapped onto $7 \cdot 8$, and (8,7) is mapped onto $8 \cdot 7$. By the commutative property of multiplication of natural numbers, we know that these two numbers are the same.

Exercise O Because of the commutative property of multiplication of natural numbers, $\underline{ab}$ is the same natural number as $\underline{ba}$ for each replacement of $\underline{a}$ and $\underline{b}$.

Exercise A N

Page 33
Answers for Exercise B For each replacement of the variables, the difference found by subtracting $\underline{b}$ from $\underline{a}$ is the same as the difference found by subtracting $\underline{a}$ from $\underline{b}$.

Exercise D Each statement is false because a number that is a member of N is never the same as a number that is not a member of N.

Exercise E No, because the condition $\underline{a} - \underline{b} = \underline{b} - \underline{a}$ is satisfied only if $\underline{a}$ and $\underline{b}$ are replaced by the same number

Exercise F One; if an operation has the commutative property, it must hold for every replacement of the variables in the condition that expresses that property.

Exercise G N

Exercise H For each replacement of the variables, the quotient found by dividing a by b is the same as the quotient found by dividing b by a.

In statement A, a has been replaced by 12, and b has been replaced by 3.

In statement B, a has been replaced by 96, and b has been replaced by 16.

In statement C, a has been replaced by 144, and b has been replaced by 24.

Exercise I No

All the statements in the display are false. $3 \neq 12$, $16 \neq 96$, and $24 \neq 144$ are not natural numbers. A number that is not a member of N is never the same as a number that is a member of N.

Exercise J No, because the condition $a \div b = b \div a$ is not satisfied by all replacements of a and b from N

Pages 33-34
Answers for "On Your Own" exercises

1	T	6	F	11	{15}	16	{0,1}
2	F	7	T	12	{5}	17	{8}
3	T	8	T	13	{0}	18	{0}
4	F	9	T	14	{13}	19	{0, 1, 2, ...}
5	F	10	T	15	{4}	20	{0, 1, 2, ...}

21 Yes; the truth of both statements follows from the definition of our operation "*."

22 No, because the number on the left is 11, and the number on the right is 10

23 No, because the number $3 * 4$ is 11, and the number $4 * 3$ is 10

24 $7 * 13 = 7 + 2(13)$ $\quad 13 * 7 = 13 + 2(7)$

$= 33.$ $\quad = 27.$

Since $33 \neq 27$, the statement $7 * 13 = 13 * 7$ is not a true statement.

25 Operation "*" is not commutative, because the condition $a * b = b * a$ is not satisfied by every replacement of the variables by members of N.

26 a has been replaced by 5, and b has been replaced by 7. The result is the number 19.

27 a has been replaced by 9, and b has been replaced by 4. The result is the number 17.

28 $\underline{a}$ has been replaced by 11, and $\underline{b}$ has been replaced by 6. The result is the number 23.

29 $\underline{a}$ has been replaced by 19, and $\underline{b}$ has been replaced by 35. The result is the number 89.

30 Examples 25-29 seem to indicate that the set of natural numbers is closed under the operation "*" since each of the selected replacements for the variables maps $(\underline{a},\underline{b})$ onto a member of N. This, of course, is not a proof.

31 $\underline{a} \,\text{↯}\, \underline{b} = 3\underline{a} + 3\underline{b}$.

Since $3\underline{a} + 3\underline{b} = 3\underline{b} + 3\underline{a}$ results in a true statement for every replacement of the variables, $\underline{a} \,\text{↯}\, \underline{b} = \underline{b} \,\text{↯}\, \underline{a}$, and the operation "↯" is commutative.

Since a member of N is obtained from $3\underline{a} + 3\underline{b}$ for each replacement of $\underline{a}$ and $\underline{b}$, the set of natural numbers is closed under the operation "↯."

32 $\underline{a} \,\text{▽}\, \underline{b} = \underline{a} + 2\underline{b} - 4$.

Since a true statement is not obtained from $\underline{a} + 2\underline{b} - 4 = \underline{b} + 2\underline{a} - 4$ for each replacement of $\underline{a}$ and $\underline{b}$, the operation "▽" is not commutative. For example, if a is replaced by 2, and b is replaced by 3, $\underline{a} + 2\underline{b} - 4 = 2 + 6 - 4$, or 4; and $\underline{b} + 2\underline{a} - 4 = 3 + 4 - 4$, or 3. $3 \neq 4$. Since a member of N is not obtained for each replacement of $\underline{a}$ and $\underline{b}$ in $\underline{a} + 2\underline{b} - 4$, the set of natural numbers is not closed under the operation "▽."

For example, if $\underline{a}$ is replaced by 3, and b is replaced by 0, the number $3 + 0 - 4$ is obtained. This is not a natural number.

LESSON 7

Pages 34-39 Notes

Addition, multiplication, subtraction, and division are defined as binary operations. The word binary implies that these operations are used with just two objects at a time. In this lesson the associative property is introduced. After students learn that addition and multiplication are associative (have the associative property), numerals like $3 + 8 + 5$ and $6 \times 4 \times 16 \times 5$ become meaningful. Subtraction and division are not associative and so numerals like $7 - 3 - 2$ and $18 \div 6 \div 3$ are not meaningful. (One exception to this will be given in a later lesson.)

First consider the numeral $3 + 8 + 5$. Since addition is a binary operation, we must first work with a pair of numbers. The numeral $(3 + 8) + 5$ indicates that the operation of addition is first applied to the numbers 3 and 8 to obtain the sum $3 + 8$, or 11. Then the operation of addition is applied to the numbers 11 and 5. This process of thinking about arranging numbers into pairs is called grouping. The associative property means that the method of grouping numbers does not affect the sum that we obtain. The numeral $(3 + 8) + 5$ expresses the same number as $3 + (8 + 5)$. Note that the associative property does not change the order in which we add numbers. The commutative property is concerned with order.

Next consider $(21 \cdot 19)\ 6$. Applying the commutative property, we obtain the true statement expressed below.

$$(21 \cdot 19)\ 6 = (19 \cdot 21)\ 6.$$

Now by applying the associative property, the following sentence expresses a true statement.

$$(19 \cdot 21)\ 6 = 19\ (21 \cdot 6).$$

An important property of the relation "equals" is also involved in this problem. The student will learn about this property later. The relation "equals" is transitive. In mathematical language, if the conditions $a = b$ and $b = c$ become true statements with certain replacements for a, b, and c, then the condition $a = c$ will become a true statement with the same replacements. The students have used this idea without knowing that it is a property of only certain relations. For example, consider the relation, "is the same age as."

If it is true that Mary is the same age as Jane, and it is also true that Jane is the same age as Sue, then you know that it is also true that Mary is the same age as Sue. The relation "is the same age as" is transitive.

For the answer to the "On Your Own" Exercise 27 the student should write sentences like A and B below. The reason that each sentence expresses a true statement is given to the right. Sentence C expresses a true statement because of the transitive property. The student is not expected to write this sentence, but he will know that sentences A and B also imply this statement.

A $(41 + 7)\ 3 = (7 + 41)\ 3$. Commutative property of addition

B $(7 + 41)\ 3 = 3\ (7 + 41)$. Commutative property of multiplication

C $(41 + 7)\ 3 = 3\ (7 + 41)$. Transitive property of "equals"

An important point in the lesson is that an example never proves a theorem, but one counterexample is sufficient to disprove it. The true statements $(7 - 5) - 0 = 7 - (5 - 0)$ and $6 \div (3 \div 1) = (6 \div 3) \div 1$ suggest that subtraction and division have the associative property. However, a property of an operation must hold for all permissible replacements of variables in the universe concerned. The statements $(7 - 5) - 1 = 7 - (5 - 1)$ and $(72 \div 12) \div 2 = 72 \div (12 \div 2)$ are false and are called counterexamples. These two counterexamples are enough to prove that subtraction and division do not have the associative property.

It may be helpful to organize this lesson for use during two class periods. On the first day, present the material concerning the associative property of addition along with a discussion of associativity for subtraction. On the second day, present the comparable material for multiplication and division. The "On Your Own" Exercises 1, 3, 7, 9, 10, 11, 13, 14, 17, 18, 22, 24, 25, 28, and 30 could be used for one assignment and the remaining exercises for another.

Page 34
Answer for Exercise B $(4,3)$ 7

Page 35
Answer for Exercise C It is the number onto which $(4,3)$ is mapped by addition.
13

Page 35 (cont.)
Answers for Exercise D (3,6)
No; in Exercise B, (4,3) was named.

Exercise E 9

Exercise F It is the number onto which (3,6) is mapped by addition.
13

Exercise G Yes
The number (4 + 3) + 6 is the same number as 4 + (3 + 6).

Exercise H

A	(14 + 8) + 5	14 + (8 + 5)
=	22 + 5	= 14 + 13
=	27.	= 27.

Since each member of sentence A is mapped by addition onto the number 27, the sentence expresses a true statement.

Similarly, show that sentence B and sentence C express true statements.

Exercise I Yes 5 6

Exercise J The sets are disjoint because they have no common members.

Exercise K $F \cup G = \{0, 1, 2, 3, a, b, c, d, e\}$.
$(F \cup G) \cup H = \{0, 1, 2, 3, a, b, c, d, e, +, -, \times, \div, <, >\}$.
The parentheses around $F \cup G$ tell us that this union should be tabulated first. Then the union of the set $F \cup G$ with H should be tabulated. The two tabulations are shown above.

Exercise L The parentheses indicate that $G \cup H$ should be tabulated first.
$G \cup H = \{a, b, c, d, e, +, -, \times, \div, <, >\}$.
$F \cup (G \cup H) = \{0, 1, 2, 3, a, b, c, d, e, +, -, \times, \div, <, >\}$.

Exercise N 15 15

Exercise O

(4 + 5) + 6	4 + (5 + 6)
= 9 + 6	= 4 + 11
= 15.	= 15.

The above shows why the two numbers are the same.

Page 36

Answers for Exercise P The two sets will be the same because they always have the same members. The tabulations in Exercises K and L above illustrate this observation.

Exercise Q $(A \cup B) \cup C$ $A \cup (B \cup C)$

Yes; this is a generalization of the idea developed in Exercises J through P above.

Exercise R A: $\underline{a}$ has been replaced by 9, $\underline{b}$ by 8, and $\underline{c}$ by 16.

B: $\underline{a}$ has been replaced by 1, $\underline{b}$ by 45, and $\underline{c}$ by 24.

C: $\underline{a}$ has been replaced by 13, $\underline{b}$ by 11, and $\underline{c}$ by 19.

Each statement is true because each is an application of the associative property of addition for natural numbers.

Exercise S $4 + 3 + 6$ is accepted as a name of a number because the way we group the numbers does not affect the sum. $(4 + 3) + 6$ and $4 + (3 + 6)$ are the same number.

Exercise B 2 and 5 10

Exercise C 80

Page 37

Answers for Exercise D 5 and 8 40

Exercise E 80

Exercise G $(2 \cdot 5)\ 8$ $2(5 \cdot 8)$

$= 10 \cdot 8$ $= 2 \cdot 40$

$= 80.$ $= 80.$

The statement is true because $(2 \cdot 5)8$ and $2(5 \cdot 8)$ are the same number.

Exercise H A $(7 \cdot 6)\ 14$ $7\ (6 \cdot 14)$

$= 42 \cdot 14$ $= 7 \cdot 84$

$= 588.$ $= 588.$

Since $(7 \cdot 6)14$ and $7(6 \cdot 14)$ are the same number, sentence A expresses a true statement.

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Answers for Exercise H (cont.) Similarly, show that sentence B and sentence C express true statements.

Exercise I

A: $\underline{a}$ has been replaced by 9, $\underline{b}$ by 12, and $\underline{c}$ by 17.

B: $\underline{a}$ has been replaced by 82, $\underline{b}$ by 13, and $\underline{c}$ by 5.

C: $\underline{a}$ has been replaced by 239, $\underline{b}$ by 4, and $\underline{c}$ by 11.

Each statement is true because each is an application of the associative property of multiplication for natural numbers.

Exercise J

$2 \cdot 5 \cdot 8$ is accepted as a name of a number because either grouping of the factors yields the same product. $(2 \cdot 5)8$ and $2(5 \cdot 8)$ are the same number.

Exercise K

Consider an example:

Let $A = \{1, 2, 3\}$,

$B = \{4,5\}$, and

$C = \{3,8\}$.

$A \times B = \{(1,4), (1,5), (2,4), (2,5), (3,4), (3,5)\}$.

$(A \times B) \times C = \{[(1,4),3], [(1,4),8], \cdots, [(3,5),8]\}$.

$B \times C = \{(4,3), (4,8), (5,3), (5,8)\}$.

$A \times (B \times C) = \{[1,(4,3)], [1,(4,8)], \cdots, [3,(5,8)]\}$.

Many one-to-one correspondences can be set up between the sets $(A \times B) \times C$ and $A \times (B \times C)$ tabulated above. We can agree, for example, to pair $[(1,4),3]$ with $[1,(4,3)]$, $[(2,4),8]$ with $[2,(4,8)]$, and so on, until all members of both sets are used in the two-way mapping.

Since $(A \times B) \times C$ can be put into one-to-one correspondence with $A \times (B \times C)$, the two sets are equivalent.

Exercise L

Suppose that A, B, and C are any three sets, and that $\underline{a} = \underline{n}(A)$, $\underline{b} = \underline{n}(B)$, and $\underline{c} = \underline{n}(C)$ become true statements after replacement. The universe for $\underline{a}$, $\underline{b}$, and $\underline{c}$ is N.

Then the following sentences will express true statements after replacement:

Page 37

Answers for Exercise L (cont.)

$\underline{n}(A \times B) = ab.$ $\underline{n}(B \times C) = bc.$

$\underline{n}[(A \times B) \times C] = (ab)c.$ $\underline{n}[A \times (B \times C)] = a(bc).$

Since $(A \times B) \times C$ and $A \times (B \times C)$ are equivalent sets, $\underline{(ab)c}$ is the same as $\underline{a(bc)}$ after replacement.

Exercise A N

Exercise B For each replacement of $\underline{a}$, $\underline{b}$, and $\underline{c}$, $(\underline{a} - \underline{b}) - \underline{c}$ will name the same number as $\underline{a} - (\underline{b} - \underline{c})$.

$\underline{a}$ has been replaced by 4, $\underline{b}$ by 3, and $\underline{c}$ by 1.

Exercise C $(4 - 3) - 1$ $4 - (3 - 1)$

$= 1 - 1$ $= 4 - 2$

$= 0.$ $= 2.$

Since $0 \neq 2$, the statement expressed in Display 10 is not true.

Page 38

Answers for Exercise D No

One counterexample is sufficient to show that subtraction of natural numbers does not have the associative property. Such an example is given in Exercise C above.

Exercise E No

If we could conclude, on the basis of one example, that a property is true for all natural numbers, we would be forced to conclude that subtraction of natural numbers has the associative property. Here is such an example.

In the condition $\underline{(a - b) - c} = \underline{a - (b - c)}$, replace $\underline{a}$ by 5, $\underline{b}$ by 2, and $\underline{c}$ by 0. We then have the statement

$$(5 - 2) - 0 = 5 - (2 - 0).$$

This is easily shown to be a true statement. It says that $3 = 3$.

Exercise F Yes; the symbol $(4 - 3) - 1$ is a name of the number 0, and the symbol $4 - (3 - 1)$ is a name of the number 2.

We should not accept the symbol $4 - 3 - 1$ as a name of a number because subtraction does not have the associative property, and the symbol does not show which subtraction should be done first.

Page 38 (cont.)
Answers for Exercise G N

Exercise H For each replacement of a, b, and c, the quotient found by dividing a by b, then dividing the result by c, is the same number as the quotient found by dividing a by the quotient of b and c.

a has been replaced by 60, b by 6, and c by 2.

Exercise I $(60 \div 6) \div 2 = 10 \div 2 = 5.$ $\qquad 60 \div (6 \div 2) = 60 \div 3 = 20.$

Since $5 \neq 20$, the statement expressed in Display 11 is not true.

Exercise J No (See Exercise K below for explanation.)

Exercise K One counterexample is sufficient to show that division of natural numbers does not have the associative property. Such an example is given in Exercise I above.

Exercise L Yes; the symbol $(60 \div 6) \div 2$ names the number 5, and the symbol $60 \div (6 \div 2)$ names the number 20.

The symbol $60 \div 6 \div 2$ cannot be accepted as the name of a number because division of natural numbers does not have the associative property, and the symbol does not show which division should be done first.

Pages 38-39
Answers for "On Your Own" exercises

1 T Associative property of addition
2 F
3 F
4 T Associative property of multiplication
5 F
6 F
7 T Commutative property of addition
8 F
9 T Commutative property of addition
10 F

Answers for "On Your Own" exercises (cont.)

11	{10}	18	{0, 1, 2,···}
12	{0}	19	{0}
13	{19}	20	{1, 2, 3,···}
14	{4, 5, 6,···}	21	(48 ÷ 12) ÷ 2
15	{0, 1, 2,···}	22	21 - (14 - 7)
16	{12}	23	54 ÷ (27 ÷ 9)
17	{11}	24	(18 - 9) - 3

25 Answered in text

26 $14(7 \cdot 9) = (14 \cdot 7)\,9$.
associative property of multiplication

$(14 \cdot 7)\,9 = 9(14 \cdot 7)$.

commutative property of multiplication

27 $(41 + 7)\,3 = (7 + 41)\,3$.
commutative property of addition

$(7 + 41)\,3 = 3\,(7 + 41)$.

commutative property of multiplication

28 $101_{two} + (11_{two} + 10_{two}) = 101_{two} + (10_{two} + 11_{two})$.
commutative property of addition

$101_{two} + (10_{two} + 11_{two}) = (101_{two} + 10_{two}) + 11_{two}$.
associative property of addition

29 $(21 \cdot 19)\,6 = (19 \cdot 21)\,6$.
commutative property of multiplication

$(19 \cdot 21)\,6 = 19\,(21 \cdot 6)$.

associative property of multiplication

30 $(24 + 18) + 12 = 12 + (24 + 18)$.
commutative property of addition

$12 + (24 + 18) = (12 + 24) + 18$.

associative property of addition

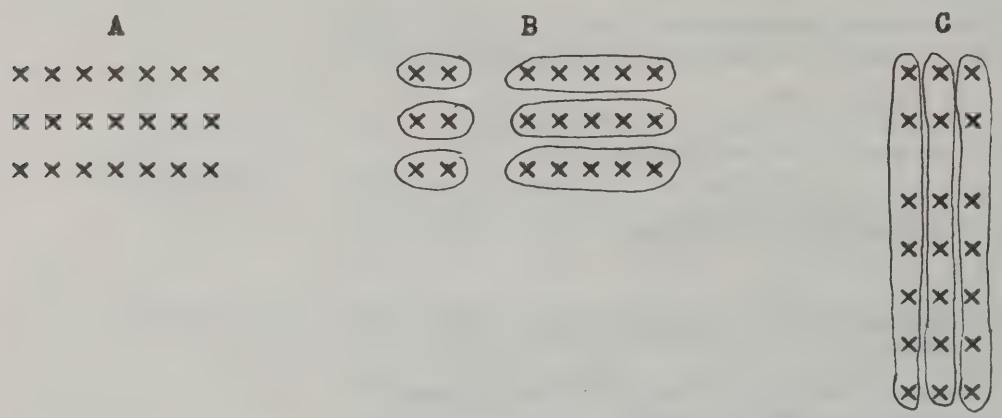
Note: When two properties are required to reach a conclusion, there is not necessarily a specific order in which they must be applied.

LESSON 8

Pages 39-45
Notes

We have defined two binary operations under which the set of natural numbers is closed. They are addition and multiplication. Each of these operations is commutative and associative. These properties have been discussed in previous lessons. The operations of addition and multiplication are linked together by means of the distributive property.

This property may be illustrated by grouping a set of objects. The number 21 is associated with each set of objects shown under A, B, and C.



The arrangement under B is indicated by the numeral $3(2) + 3(5)$. The arrangement under C is indicated by the numeral $3(2 + 5)$. The true statement $3(2 + 5) = 3(2) + 3(5)$ suggests that multiplication distributes over addition. Multiplication and addition of natural numbers are always related in this way. For each replacement of a, b, and c in the universe N, a true statement is always obtained from $a(b + c) = ab + ac$. This property is the distributive property of multiplication over addition for natural numbers.

By means of the property the student will better understand the computational processes of arithmetic. For example, the student is familiar with the computation shown below under A and B. The sentences under C show how the distributive property explains this procedure.

A	B	C
26	26	$26(34) = 26(30 + 4)$.
34	34	$26(30 + 4) = 26(30) + 26(4)$.
104	24	
780	80	$26(30) + 26(4) = (20 + 6)30 + (20 + 6)4$.
884	180	$(20 + 6)30 + (20 + 6)4 = (20)30 + (6)30 + (20)4 + (6)4$.
	600	$(20)30 + (6)30 + (20)4 + (6)4 = 600 + 180 + 80 + 24$.
	884	$600 + 180 + 80 + 24 = 884$.

The distributive property may be proved by the union of Cartesian sets of ordered pairs. For teachers who may be interested, the proof is included in Part Two of this guide.

Mathematicians speak of a left-hand and right-hand distributive property. They regard the condition $(b + c)a = ba + ca$ as distinct from the condition $a(b + c) = ab + ac$. Each of these conditions follows very easily from the other by applying the commutative property of multiplication.

After studying the distributive property of multiplication over addition, students will investigate whether or not addition distributes over multiplication. If it does, then for each replacement of a, b, and c, from N, $a + (bc) = (a + b)(a + c)$ yields a true statement. Selecting 5 as a replacement for a, 2 for b, and 3 for c, we obtain $5 + (2 \cdot 3) = (5 + 2)(5 + 3)$. Since $11 \neq 56$, addition does not distribute over multiplication. Remember, one counterexample is sufficient to disprove this property because a property of operations must hold without exception.

Multiplication does, however, distribute over subtraction. To test distribution of multiplication over subtraction, use the condition $a(b - c) = ab - ac$. Any replacements for a, b, and c that the students may suggest will yield true statements so long as the replacement for b is greater than the replacement for c. An example, of course, is not a proof, but several examples are usually sufficient to convince students that a general rule is justified.

To determine whether or not subtraction distributes over multiplication, use the condition $a - (bc) = (a - b)(a - c)$. To determine whether or not division distributes over addition, use the condition $a \div (b + c) = (a \div b) + (a \div c)$. Other comparable investigations could be made. Some are suggested

in the text on page 44.

The students may fail to see the general distributive property because of their familiarity with the ordinary signs of operation. Use the condition $a \square (b \triangle c) = (a \square b) \triangle (a \square c)$. Have the students replace $\square$ and $\triangle$ by $\times$, $+$, $-$, or $\div$. The replacements for $\square$ and $\triangle$ maybe the same or different. Then, ask if a true statement is obtained for any meaningful replacement of a, b, and c from N. For example, replace $\square$ by $+$ and $\triangle$ by $\div$. The condition now becomes $a + (b \div c) = (a + b) \div (a + c)$. Examples of statements obtained are: $8 + (24 \div 8) = (8 + 24) \div (8 + 8)$, $4 + (6 \div 1) = (4 + 6) \div (4 + 1)$, and $2 + (10 \div 2) = (2 + 10) \div (2 + 2)$. This proves that addition does not distribute over division.

The content of the lesson from page 39 through 43 can probably be taught during one class session. You may wish to omit page 44 or use it as a supplementary project for some of the students. Give careful attention to the illustrative examples on pages 39-41 so that the meaning of the distributive property will be made clear. To answer Exercise H on page 43, the student should use Display 17 and his intuitive understanding of the transitive property of "equals."

Page 39

Answers for Exercise B — There are 2 groups of 4 albums. The albums in one group each have 2 records, and the albums in the other group each have 3 records.

Exercise C — Yes; the action requires that 1 group of 4 albums, each containing 2 records, be combined with another group of 4 albums, each containing 3 records.

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Answers for Exercise D — Yes; $4(2) + 4(3)$ and $\underline{8+12}$ are the same number. {20}

Exercise F — There will be 4 groups of $\underline{2+3}$ records each.

Exercise H — Yes; the left member expresses 4 groups of objects with $\underline{2+3}$ objects in each group.

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Answers for Exercise H (cont.) — x is a variable for the number of records that Jean has.

Exercise K — Yes; $4(2 + 3)$ is the same number as $4(2) + 4(3)$.

Page 41

Answers for Exercise N — Yes; it expresses the requirement that the perimeter of the field include 2 lengths of 100 yards and 2 lengths of 80 yards.

x is a variable for the total number of yards in the perimeter of the field.

Exercise O — Yes; $2(100) + 2(80)$ is the same number as $200 + 160$.

{360}

Exercise R — Yes

x is a variable for the total number of yards in the perimeter of the field.

Exercise S — Yes; $(100 + 80)$ is the same number as 180.

{360}

Exercise U — The statement is true because $2(100 + 80)$ is the same number as $2(100) + 2(80)$.

Exercise A — 6

Sentence A expresses a true statement because $6(5) + 6(9)$ is the same number as $6(5 + 9)$.

Exercise B — 3 and 7

10

Page 42

Answers for Exercise C — $9(18) + 9(4) = 9(18 + 4)$.

Exercise D — Since $2(13 + 1)$ is the same as $2(13) + 2(1)$, sentence A expresses a true statement. Similarly, sentences B and C also express true statements.

Exercise E: — One way: Add the number replacing b and the number replacing c; then multiply the sum by the number replacing a.

Page 42

Answers for Exercise E (cont.) Other way: Multiply the number replacing b by the number replacing a; multiply the number replacing c by the number replacing a; add the two products.

Answers for Exercise F In the statement expressed by sentence A, a has been replaced by 3, b by 14, and c by 9.

In the statement expressed by sentence B, a has been replaced by 16, b by 12, and c by 8.

In the statement expressed by sentence C, a has been replaced by 41, b by 37, and c by 23.

In the statement expressed by sentence D, a, b, and c have each been replaced by 3.

Each statement is true by the distributive property of multiplication over addition for natural numbers.

Exercise H Yes; 6×53 is the same number as $6(50 + 3)$.

Exercise I Yes; upon the distributive property of multiplication over addition for natural numbers

Page 43

Answers for Exercise J Yes; $6(50)$ is the same number as 300; $6(3)$ is the same number as 18; and $300+18$ is the same number as 318.

Exercise K

a $8 \times 47 = 8(40 + 7)$.
$8(40 + 7) = 8(40) + 8(7)$.
$8(40) + 8(7) = 320 + 56$.
$320 + 56 = 376$.

b $7 \times 92 = 7(90 + 2)$.
$7(90 + 2) = 7(90) + 7(2)$.
$7(90) + 7(2) = 630 + 14$.
$630 + 14 = 644$.

c $79 \times 3 = 3 \times 79$.
$3 \times 79 = 3(70 + 9)$.
$3(70 + 9) = 3(70) + 3(9)$.
$3(70) + 3(9) = 210 + 27$.
$210 + 27 = 237$.

Page 43

Answers for Exercise K (cont.)

d $35 \times 26 = 35(20 + 6)$.

$35(20 + 6) = 35(20) + 35(6)$.

$35(20) + 35(6) = 20(35) + 6(35)$. (Commutative law for natural numbers is applied)

$20(35) + 6(35) = 20(30 + 5) + 6(30 + 5)$.

$20(30 + 5) + 6(30 + 5) = 20(30) + 20(5) + 6(30) + 6(5)$.

$20(30) + 20(5) + 6(30) + 6(5) = 600 + 100 + 180 + 30$.

$600 + 100 + 180 + 30 = 910$.

Exercise A Given any 3 natural numbers; if we multiply the sum of the second and third by the first, the result is the same as if we add the product of the third and first to the product of the second and first.

Exercise B It differs only in form.

Exercise C In the statement expressed by sentence A, a has been replaced by 13, b by 6, and c by 8.

In the statement expressed by sentence B, a has been replaced by 4, b by 15, and c by 19.

In the statement expressed by sentence C, a has been replaced by 83, b by 7, and c by 5.

Each statement is true. The truth of each statement can be proved by carrying through the indicated computations.

Exercise D Yes - For reason, see Exercise E.

Exercise E The commutative property of multiplication for natural numbers

Exercise F Yes; upon the distributive property of multiplication over addition for natural numbers

Exercise G Yes; upon the commutative property of multiplication for natural numbers

Exercise H Yes

Display 17 shows the steps for explaining this.

Page 43 (cont.)

Answers for Exercise A Give any three natural numbers; if we add the first and the product of the second and the third, the result is the same as if we multiply the sum of the first and third by the sum of the first and second.

Exercise B a has been replaced by 3, b by 4, and c by 7.

Page 44

Answers for Exercise C Sentence A expresses a true statement, since $4 \cdot 7$ is the same number as 28.

Sentence B expresses a true statement, since $(3 + 4)(3 + 7)$ is the same number as $7 \cdot 10$.

Exercise D No, because $3 + (4 \cdot 7)$ is not the same number as $(3 + 4)(3 + 7)$

Exercise E Exercises C and D above show that addition does not distribute over multiplication for natural numbers.

Exercise F If the condition expressed in Display 18 does not yield a true statement for all replacements from N, then the condition is not a property for N.

Exercise G Given any three natural numbers; if we subtract the third from the second and multiply the difference by the first, the result is the same as if we subtract the product of the first and the third from the product of the first and the second.

Exercise H a has been replaced by 5, b by 6, and c by 2.

Exercise I The following computation shows that the statement is true:

$$5(6 - 2) = 5(4). \qquad 5(6) - 5(2) = 30 - 10.$$
$$5(4) = 20. \qquad 30 - 10 = 20.$$

Exercise J There are many correct answers. The following are typical:

$$7(13 - 9) = 7(13) - 7(9).$$
$$16(11 - 8) = 16(11) - 16(8).$$

Exercise K The 3 examples do not prove or disprove, but suggest, that multiplication distributes over subtraction for all natural numbers.

Page 44

Answers for Exercise K (cont.) Remember, however, that N is not closed under subtraction, so such a distributive property would apply only when the differences are natural numbers.

Exercise L

A Given any three natural numbers; if we find the product of the second and third and subtract this product from the first, the result is the same as if we multiply the difference, found by subtracting the third from the first, by the difference, found by subtracting the second from the first.

B Given any three natural numbers; if we subtract the third from the second and add this difference to the first, the result is the same as if we subtract the sum of the first and third from the sum of the first and second.

C Given any three natural numbers; if we subtract the sum of the second and third from the first, the result is the same as if we subtract the second from the first, subtract the third from the first, and then find the sum of these differences.

D Given any three natural numbers; if we divide the second by the third and find the sum of this quotient and the first, the result is the same as if we divide the sum of the first and second by the sum of the first and third.

Exercise M There are many sentences that could be written in response to the instruction. The following are typical.

A $18 - (2 \cdot 3) = (18 - 2)(18 - 3)$.

B $9 + (10 - 3) = (9 + 10) - (9 + 3)$.

C $8 - (2 + 3) = (8 - 2) + (8 - 3)$.

D $6 + (10 \div 2) = (6 + 10) \div (6 + 2)$.

Exercise N None of the sentences expresses a true statement.

Exercise O Since a false statement is obtained from sentence A in Display 21, subtraction does not distribute over multiplication for natural numbers.

Page 44
Answers for Exercise O (cont.)

Since a false statement is obtained from sentence B in Display 21, addition does not distribute over subtraction for natural numbers.

Since a false statement is obtained from sentence C in Display 21, subtraction does not distribute over addition for natural numbers.

Since a false statement is obtained from sentence D in Display 21, addition does not distribute over division for natural numbers.

Pages 44-45
Answers for "On Your Own" exercises

1	T	Distributive property of multiplication over addition
2	T	Commutative property of addition
3	T	Associative property of multiplication
4	F	
5	T	Distributive property of multiplication over addition
6	T	Distributive property of multiplication over subtraction
7	T	Associative property of addition
8	T	Commutative property of multiplication
9	F	
10	T	Commutative property of addition
11	T	Distributive property of multiplication over subtraction
12	F	
13	T	Commutative property of multiplication
14	F	
15	{4}	21 {2, 3, 4, ...}

Answers for "On Your Own" exercises (cont.)

16 {0, 1, 2, 3}

17 {2}

18 {0, 1, 2, •••}

19 {57}

20 {0, 1, 2, •••}

22 {16}

23 {69}

24 {0, 1, 2, •••, 16}

25 {0, 1, 2, •••}

26 {0, 1, 2, •••}

Note: Several of the exercises, 27-36, have more than one correct solution.

27 $15 + 35 = 5(3) + 5(7)$.
$5(3) + 5(7) = 5(3 + 7)$.

28 $32 + 96 = 8(4) + 8(12)$.
$8(4) + 8(12) = 8(4 + 12)$.

29 $56 + 14 = 7(8) + 7(2)$.
$7(8) + 7(2) = 7(8 + 2)$.

30 $45 + 18 = 9(5) + 9(2)$.
$9(5) + 9(2) = 9(5 + 2)$.

31 $168 + 24 = 8(21) + 8(3)$.
$8(21) + 8(3) = 8(21 + 3)$.

32 $22 + 121 = 11(2) + 11(11)$.
$11(2) + 11(11) = 11(2 + 11)$.

33 $75 + 125 = 25(3) + 25(5)$.
$25(3) + 25(5) = 25(3 + 5)$.

34 $130 + 20 = 10(13) + 10(2)$.
$10(13) + 10(2) = 10(13 + 2)$.

35 $16 + 34 = 2(8) + 2(17)$.
$2(8) + 2(17) = 2(8 + 17)$.

36 $30 + 75 = 15(2) + 15(5)$.
$15(2) + 15(5) = 15(2 + 5)$.

LESSON 9

Pages 45-48
Notes

There is a distinction between operation and computation. An operation is performed on numbers, while computation is the processing of numerals. A binary operation assigns an ordered pair of numbers to a number. (3,2) is assigned to 3+2 by addition. In making this assignment we have carried out the operation of addition. A numeral, such as 3 + 2, that contains one or more symbols for operations is an expression. A standard name of a natural number is one of the numerals in the list 0, 1, 2,···. To simplify an expression is to find the standard name for the number that it represents.

In written English specific conventions are observed because clarity of communication is necessary. Note the difference between a high school-building and a high-school building. The position of the hyphen changes the idea that the words express. Similarly, in mathematics we agree to use certain symbols in specific ways. The purpose is to avoid contradictions that arise from differing interpretations.

The meaning of parentheses in expressions is quite clear. The expression $5 + (6 \cdot 2)$ names the same number as the expression $5 + 12$. Now consider the expression $5 + 6 \cdot 2$. Is $5 + 6 \cdot 2$ a numeral for the number 22 or the number 17? The answer depends upon the order in which the operations of addition and multiplication have been performed. We agree that the numeral $5 + 6 \cdot 2$ indicates that multiplication has been performed first, then addition. So $5 + 6 \cdot 2 = 17$ is a true statement, but $5 + 6 \cdot 2 = 22$ is not a true statement. The general rule that may be applied is that, to simplify expressions not containing parentheses, you find the standard name of all products and quotients before finding the standard name of sums and differences. The following sentences express true statements:

$$13 - 5 \cdot 2 = 13 - 10. \qquad 15 \div 3 + 2 = 5 + 2.$$
$$13 - 10 = 3. \qquad 5 + 2 = 7.$$

Parentheses are still necessary in some expressions. The "On Your Own" exercises illustrate this. This lesson can be covered in one class session.

Page 45
Answers for Exercise A

B We have obtained the number $\underline{8 \cdot 2}$ by multiplication.

C We have obtained the number $\underline{19-3}$ by subtraction.

D We have obtained the number $\underline{32 \div 2}$ by division.

Page 46
Answers for Exercise C

No, because the numeral 32 ÷ 2 does not occur in the list of symbols 0, 1, 2, ···. 32÷2 contains a symbol for division so 32÷2 is an expression.

Exercise D

The standard names 8 and 2; the operation symbol for times

Exercise E

A 16 C 16

B 16 D 16

Exercise F

Multiplication of 4 and 5; multiplication of 4 and 3; addition of 4(5) and 4(3)

Exercise G

32

Exercise H

Yes, because $\underline{4(5) + 4(3)}$ is the same number as 32

Exercise I

The symbol '( )' for multiplication, and the symbol '-' for subtraction

Exercise J

$6(7) - 6(2) = 42 - 12.$
$42 - 12 = 30.$

Exercise K

The symbol '+' for addition, and the symbol '·' for multiplication

Exercise L

$(4 + 3)6 = 7(6).$
$7(6) = 42.$

Page 47
Answers for Exercise M

$4 + (3 \cdot 6) = 4 + 18.$
$4 + 18 = 22.$

Exercise N

No; the standard name obtained in Exercise M is 22, while the standard name obtained in Exercise L is 42.

Exercise O

18

Exercise P

22

Page 47 (cont.)

Answers for Exercise Q The statement is true because $4 + 3 \cdot 6$ is the same number as 22.

Exercise R In expressions that do not include parentheses, we have decided to divide first and then add. Therefore, $9 + 24 \div 4$ means that 24 has been divided by 4, and the quotient has been added to 9.

$9 + 24 \div 4 = 9 + 6.$
$9 + 6 = 15.$

Exercise S

B $4 \cdot 2 + 7 = 8 + 7.$
$8 + 7 = 15.$

C $25 - 5 \cdot 2 = 25 - 10.$
$25 - 10 = 15.$

D $3 \cdot 2 + 3 \cdot 3 = 6 + 9.$
$6 + 9 = 15.$

E $42 \div 2 - 18 \div 3 = 21 - 6.$
$21 - 6 = 15.$

F $3 \cdot 4 + 12 \div 4 = 12 + 3.$
$12 + 3 = 15.$

Exercise T Yes; they are names for the number 15.

Pages 47-48

Answers for "On Your Own" exercises

Note: The number of available names for a given number is infinite.

1 Some typical names of 6 are $18 \div 3$, $10 - 4$, $2 + 4$, $2(3)$, and $2(1 + 2)$.

2 Some typical names of 18 are $2 \cdot 9$, $9 + 9$, $3(3 + 3)$, $3(11 - 5)$, $54 \div 3$.

3 Some typical names of 25 are $48 - 23$, $100 \div 4$, $5 \cdot 5$, $150 \div 2 - 50$, $12 + 13$.

4 Some typical names of 27 are $3 \cdot 3 \cdot 3$, $3(2 + 7)$, $3(21 - 12)$, $8 \cdot 6 - 21$, $5 \cdot 7 - 32 \div 4$.

5 $9 + 6 \cdot 2 = 9 + 12.$
$9 + 12 = 21.$

6 $18 \div 6 + 7 = 3 + 7.$
$3 + 7 = 10.$

11 $6(5+2) + 4 \cdot 3 = 6(7) + 4 \cdot 3.$
$6(7) + 4 \cdot 3 = 42 + 12.$
$42 + 12 = 54.$

12 $7 + 8 \cdot 3 + 6 = 7 + 24 + 6.$
$7 + 24 + 6 = 37.$

Answers for "On Your Own" exercises (cont.)

7 $8 \cdot 9 + 8 \cdot 3 = 72 + 24.$
$72 + 24 = 96.$

8 $16 - 5 \cdot 3 = 16 - 15.$
$16 - 15 = 1.$

9 $13 - 24 \div 8 = 13 - 3.$
$13 - 3 = 10.$

10 $(2 + 3)(2 + 7) = 5 \cdot 9.$
$5 \cdot 9 = 45.$

13 $12 \cdot 5 - 45 \div 3 = 60 - 15.$
$60 - 15 = 45.$

14 $16 + (6 + 2)5 = 16 + 8(5).$
$16 + 8(5) = 16 + 40.$
$16 + 40 = 56.$

15 $8(4 + 6)(3 + 7) = 8 \cdot 10 \cdot 10.$
$8 \cdot 10 \cdot 10 = 800.$

16 T
17 F
18 F
19 T
20 T
21 F
22 F
23 T
24 T
25 F
26 F
27 T

28 $(2 \cdot 5) + 6 = 16.$
29 $13 - (8 - 3) = 8.$
30 $(5 + 4)(3 + 1) = 36.$
31 $60 = 2(7 - 2) \cdot 6.$
32 $(36 \div 9) + (3 \cdot 7) = 25.$
33 $3(6 + 4) - 2 = 28.$
34 $9 = (14 \div 2) + (3 - 1).$
35 $(6 + 3)(2 + 4) = 54.$
36 $8 = (18 + 2) \div 5 + 4.$
37 $(42 \div 6) - (30 \div 5) = 1.$
38 $10 = (40 \div 5) + 3 - 1.$

LESSON 10

Pages 48-51 Notes

In this lesson, the natural number system is outlined in Display 1, page 49. The natural number system is made up of the set of natural numbers and two operations, addition and multiplication.

Each of the operations is closed, commutative, and associative for the set of natural numbers; and the two operations are linked together by the distributive property of multiplication over addition.

In mathematical literature, there is a slight variation among the definitions of a number system. All the definitions include a set of objects and two operations. Some writers include the properties of 0 and 1. A later lesson is devoted to the properties of 0 and 1. The definition adopted in this book is the easiest for the student to understand.

There are many number systems in mathematics. At first thought, it would seem that any proper subset of N would satisfy the requirements of a number system, but this is not the case. It is true that addition and multiplication are associative, commutative, and distributive in any subset of N. This is true because every member of a subset of N is also a member of N. If a counterexample could be found to show that addition and multiplication are not commutative, associative, and distributive in a subset of N, then this example would also show that the two operations are not commutative, associative, and distributive in N. Consider {0, 2, 4, 6, ...}. If true statements were not obtained from all replacements of $\underline{a}$ and $\underline{b}$, using this set as the universe, in the condition $\underline{a} + \underline{b} = \underline{b} + \underline{a}$, we would have some members of N that are not commutative under the operation of addition. This follows because every member of {0, 2, 4, 6, ...} is also a member of N. Consider the numbers onto which the members of the set of odd numbers are mapped by addition. They are even numbers. So the set of odd numbers is not closed under addition. Therefore, it does not qualify as a number system. This reasoning leads to the conclusion that, in order to decide whether or not a particular subset of N is a number system, we need consider only whether or not it is closed under addition and multiplication. The other properties are present because the members of the subset are members of N.

Drawing conclusions like those above is new to the students, so the truth of the statements in Displays 3, 4, and 5 should be established by carrying through the computations indicated. The purpose of Exercises R and S on page 50 is to call attention to the generalizations made above.

Page 48

Answers for Exercise A — The set of natural numbers

Exercise B — Closure property of addition: $a + b$ is a member of N.
Commutative property of addition: $a + b = b + a$.
Associative property of addition: $(a + b) + c = a + (b + c)$.

Closure property of multiplication: ab is a member of N.
Commutative property of multiplication: $ab = ba$.
Associative property of multiplication: $(ab)c = a(bc)$.

Exercise C — Distributive property: $a(b + c) = ab + ac$.

Exercise D
a Associative property of multiplication
b Distributive property
c Commutative property of addition
d Associative property of addition
e Commutative property of multiplication
f Closure property of multiplication
g Closure property of addition

Page 49

Answers for Exercise A — Yes Yes
The set of even natural numbers
Note: Any natural number that has a factor of 2 is an even natural number.

Exercise B — 2 is a factor of 6 and of 14. 6 and 14 are, therefore, even numbers.

Exercise C — $6 + 14 = 2(3 + 7)$ is a true statement, so $6+14$ is an even number.

Exercise E — $6 \times 14 = 2(3 \cdot 14)$ is a true statement, so 6×14 is an even number.

Exercise F — Results of Exercises C and E above suggest that E is closed under addition and multiplication. For each replacement used, an even natural number was obtained from $a + b$ and from ab.

Exercise G — No; a proof of closure must show that the property holds for every possible case. Examples illustrate a property but they do not prove it.

Page 49 (cont.)

Answers for Exercise H

Since $2m$ and $2n$ each represent any even number, $2m + 2n$ represents the sum of any two even numbers.

By the distributive property, $2m + 2n = 2(m + n)$ is a true statement for every replacement of the variables. Therefore, 2 is a factor of the sum and the sum is an even number.

Exercise I

If $2m$ and $2n$ each represent any even number, $2n \cdot 2m$ represents the product of any two even numbers. 2 is a factor of this product, and therefore, the product is an even number.

Exercise J

We can say this, because we have shown that:

a The sum of two even numbers is always a member of the set of even numbers.

b The product of two even numbers is always a member of the set of even numbers.

Exercise K

A: a has been replaced by 2, and b by 8.

B: a has been replaced by 22, and b by 34.

C: a has been replaced by 74, and b by 18.

D: a has been replaced by 2, and b by 8.

E: a has been replaced by 22, and b by 34.

F: a has been replaced by 74, and b by 18.

Each statement is true.

Page 50

Answers for Exercise L

The examples in Exercise K suggest that E has the commutative property of addition and the commutative property of multiplication.

Exercise M

A: a has been replaced by 6, b by 12, and c by 30.

B: a has been replaced by 14, b by 2, and c by 28.

C: a has been replaced by 18, b by 4, and c by 10.

D: a has been replaced by 32, b by 20, and c by 8.

Each statement is true.

Exercise N

The examples is Display 4 suggest that E has the associative property of addition and the associative property of multiplication.

Answers for "On Your Own" exercises (cont.)

5 F

6 T Commutative property of multiplication

7 T Associative property of addition

8 T Closure property of addition

9 F

10 T Commutative property of addition

11 T Commutative property of multiplication

12 F

13 T Closure property of multiplication

14 T Associative property of addition

15 T Associative property of multiplication

16 {5}

17 {5}

18 {3}

19 {0, 1, 2, 3}

20 {8}

21 {0, 1, 2, ···}

22 {0, 1, 2, ···}

23 {0, 1, 2, ···}

24 {0}

25 {1, 3, 5, ···} is not a number system relative to addition and multiplication. It does not have the closure property of addition.

Page 50 (cont.)
Answers for Exercise O

A: a has been replaced by 4, b by 12, and c by 8.

B: a has been replaced by 2, b by 6, and c by 10.

C: a has been replaced by 56, b by 24, and c by 18.

Each statement is true.

Exercise P — The examples in Exercise O suggest that multiplication distributes over addition in the set of even numbers.

Exercise Q — The set of even natural numbers is a number system relative to addition and multiplication because each of these operations has a closure property, a commutative property, and an associative property; and the two operations are linked together by the distributive property of multiplication over addition.

Exercise R — If a subset of N did not have the commutative property of addition, then neither would N. The same is true of the commutative property of multiplication, the associative properties, and the distributive property. Since these properties do hold for N, they must hold for any subset of N in which addition and multiplication are defined.

Exercise S — Yes; the property of closure

As noted in Exercise R, the commutative properties, the associative properties, and the distributive property hold for subsets of N because they hold for N. However, the same conclusion is not warranted for closure.

Consider the example $\{1, 3, 5, \cdots\}$. It is not closed under addition because the sum of any two of its members is an even number instead of an odd number.

Pages 50-51
Answers for "On Your Own" exercises

1 T Commutative property of addition

2 T Associative property of multiplication

3 T Distributive property of multiplication over addition

4 T Closure property of addition

Answers for "On Your Own" exercises (cont.)

26 $\{4, 8, 12, \cdots\}$ is a number system relative to addition and multiplication.

28 $\{3, 5, 6, 9, 10, 12, 15, 19, 20, \cdots\}$ is not a number system relative to addition and multiplication. Since $\underline{3 + 5}$ is not a member of the set, it does not have the closure property of addition.

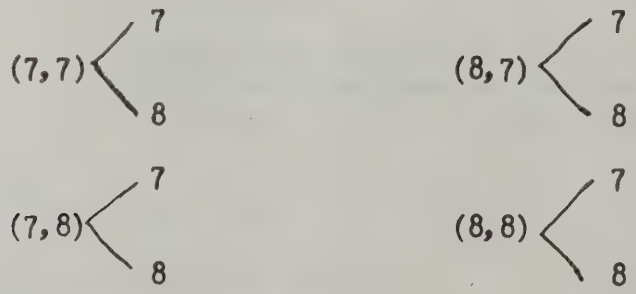
LESSON 11

Pages 51-60 Notes

If the set in a number system is an infinite set, all possible statements cannot be tested for a given property. However, if the set in a number system is a finite set, the students can test every possible case. The number of possible cases follows an interesting pattern. Suppose that the set under examination has two members. To prove that addition is commutative requires only four statements. This is because, in the condition $a + b = b + a$, there are only two possible replacements for a. Then for each replacement of a, there are only two possible replacements for b. Consider $\{7, 8\}$ and commutativity under addition as defined for set N. The pattern below shows how to form the four sentences in question.

The sentences are: $7 + 7 = 7 + 7$, $7 + 8 = 8 + 7$, $8 + 7 = 7 + 8$, and $8 + 8 = 8 + 8$.

To test the associative property, the sentences must be of the form $(a + b) + c = a + (b + c)$. If the set is $\{7, 8\}$, there are 8 sentences. This is true because exactly four different ordered pairs can be used to replace a and b. Then for each of these four replacements of a and b, exactly two replacements can be made for c.



The sentences are:

$(7 + 7) + 7 = 7 + (7 + 7)$.
$(7 + 7) + 8 = 7 + (7 + 8)$.

$$(7 + 8) + 7 = 7 + (8 + 7).$$
$$(7 + 8) + 8 = 7 + (8 + 8).$$
$$(8 + 7) + 7 = 8 + (7 + 7).$$
$$(8 + 7) + 8 = 8 + (7 + 8).$$
$$(8 + 8) + 7 = 8 + (8 + 7).$$
$$(8 + 8) + 8 = 8 + (8 + 8).$$

In the test for distribution of multiplication over addition for a set of two members, 8 sentences of the form $\underline{a}(\underline{b} + \underline{c}) = \underline{a} \cdot \underline{b} + \underline{a} \cdot \underline{c}$ are needed.

If the set being tested has three members, 9 sentences are needed to test the commutative property, 27 to test the associative property, and 27 to test the distributive property. For 4 members in the set, 16, 64, and 64 sentences are needed respectively. In general, if the set contains $\underline{n}$ members, then $\underline{n}^2$, $\underline{n}^3$, and $\underline{n}^3$ sentences are needed.

The first number system considered in the lesson is {E,O}. E is {0, 2, 4, 6, ...} and O is {1, 3, 5, 7, ...}. But all the tests can be carried through without tabulation of E and O. To set up the addition and multiplication tables, all one needs to consider is whether or not the sum or product is an even number or an odd number. Since the only possible ordered pairs of E and O are (E,E), (E,O), (O,E), and (O,O), the addition table will include the names of 4 sums, and the multiplication table will include the names of 4 products. Four sentences will be needed to test the commutative property, and 8 to test the associative property of each operation. To determine whether or not multiplication distributes over addition, 8 sentences are required. The tables and most of the necessary sentences are given in the text.

The content of the lesson from pages 51-54, through Exercise J, has to do with proving that {E,O} along with the operations of addition and multiplication satisfy the requirements of a number system. This material can be covered in one class session.

The next number system considered in the lesson is a <u>remainder system</u>. In remainder systems, a number, called the <u>modulus</u>, is chosen as a divisor for members of N. Suppose the members of a set S are 0, 1, 2, 3, 4, 5, 6, 7,

8, 9, 10, 11. S is the set of possible remainders obtained by dividing the members of N by 12. 12 is the modulus of set S.

The set of possible remainders for the number 6 has 6 members. They are 0, 1, 2, 3, 4, and 5. Call the set of these remainders T. T = {0, 1, 2, 3, 4, 5}. In this system $3 + 4 = 1$ is a true statement because $7 \div 6$ yields a quotient of 1 and a remainder of 1. So 1 is the sum of 3 and 4. Also $3 \times 4 = 0$ is a true statement because $12 \div 6$ yields a quotient of 2 and a remainder of 0. So 0 is the product of 3 and 4. The addition table for this system names 36 sums. The multiplication table names 36 products.

In teaching number systems that use sets of remainders, each of the required properties should be tested enough times to convince the students that these properties are valid for the system.

Remainder systems, in contrast to the system of natural numbers, are closed under subtraction. By definition, (3,4) is mapped onto the number that satisfies $3 = x + 4$. Using a modulus of 6, $3 = x + 4$ is satisfied by 5. In other words, $3 = 5 + 4$ is a true statement. Thus, in a remainder system that uses a modulus of 6, (3, 4) is mapped by subtraction onto 5. The remainder system that uses a modulus of 6 is not closed under division.

The last system considered is an <u>abstract</u> system. The set has two members designated <u>c</u> and <u>d</u>. Two binary operations $\ast$ and $\#$ are defined by the given tables. Using these tables, {<u>c</u>, <u>d</u>} and $\ast$ and $\#$ can be tested for the necessary properties of a number system. This system is not a number system.

Page 51
Answers for Exercise A Two

No; it has a finite number of members.

Exercise B The product of any two even numbers is an even number.

Page 52
Answers for Exercise D 0

Exercise E 0 and E

Page 52 (cont.)
Answers for Exercise F E

Exercise G E and O

E and E

Exercise I Set M is closed under both operations since the sum of any two members of M is a member of M, and the product of any two members of M is a member of M.

Exercise K O O

Exercise L a was replaced by E, and b was replaced by O.

Yes, because E + O and O + E are the same sum

Exercise M Sentence B expresses a true statement because E + E and E + E are the same sum.

Sentence C expresses a true statement because O + O and O + O are the same sum.

Sentence D expresses a true statement because O + E and E + O are the same sum.

Exercise N The product of E and O is named in the table under E and to the right of O. This product is E.

The product of O and E is named in the table under O and to the right of E. This product is E.

Since the two products are the same, the sentence E × O = O × E expresses a true statement.

a was replaced by E, and b was replaced by O.

Exercise O Sentence F expresses a true statement because E × E and E × E are the same product.

Sentence G expresses a true statement because O × O and O × O are the same product.

Sentence H expresses a true statement because O × E and E × O are the same product.

Exercise P Yes

Since there are only two rows and two columns in each table, only four different statements can be derived from each table.

Page 52 (cont.)

Answers for Exercise Q — Since all the sentences in Display 4 express true statements, and since these sentences express all possible statements obtained from $a + b = b + a$ and $ab = ba$ when the members of set M are used as replacements for the variables, both addition and multiplication are commutative in set M.

Exercise A — O is the sum of E and O because it is below E and to the right of O in the addition table.

Exercise B — The development in Exercise A above shows that the sum $(E + O) + E$ is O.

Page 53

Answers for Exercise C — O is the sum of O and E because O is below O and to the right of E in the addition table.

Exercise E — The development in Exercises C and D shows that the sum $E + (O + E)$ is O.

Exercise F — Yes, because $(E + O) + E$ and $E + (O + E)$ are the same member of M

Exercise G — a was replaced by E, b by O, and c by E.

Exercise H — Each of these statements can be checked by the method used to prove the truth of the statement expressed by sentence A.

Exercise J — Yes

The sentences in Display 5 express true statements. Since these sentences express all statements that can be obtained from $(a + b) + c = a + (b + c)$, when the members of set M are used as replacements for the variables, the operation of addition is associative in set M.

Exercise K — $(E \times O)\ E = E \times E.$ $\qquad E\ (O \times E) = E \times E.$

$E \times E = E.$ $\qquad E \times E = E.$

Exercise L — The truth of both statements is evident from Exercise K above.

Exercise M — Yes; because $(E \times O)E$ and $E(O \times E)$ are the same member of M

Page 53 (cont.)

Answers for Exercise N — $\underline{a}$ was replaced by E, $\underline{b}$ by O, and $\underline{c}$ by E.

Exercise O

$(O \times E)O = O(E \times O)$.
$(E \times O)O = E(O \times O)$.
$(O \times E)E = O(E \times E)$.
$(E \times E)O = E(E \times O)$.
$(O \times O)E = O(O \times E)$.
$(E \times E)E = E(E \times E)$.
$(O \times O)O = O(O \times O)$.

Exercise P — Yes

Each of these statements can be checked by the method shown in Exercise K.

Exercise Q — Yes

All the sentences in Exercises M and O above express true statements. Since these sentences express all statements that can be obtained from $(\underline{ab})\underline{c} = \underline{a}(\underline{bc})$, when the members of set M are used as replacements for the variables, the operation of multiplication is associative in set M.

Exercise A

A	$E(E + O) = E \times O$.	$(E \times E) + (E \times O) = E + E$.
	$E \times O = E$.	$E + E = E$.
B	$O(E + O) = O \times O$.	$(O \times E) + (O \times O) = E + O$.
	$O \times O = O$.	$E + O = O$.
C	$E(O + O) = E \times E$.	$(E \times O) + (E \times O) = E + E$.
	$E \times E = E$.	$E + E = E$.
D	$O(E + E) = O \times E$.	$(O \times E) + (O \times E) = E + E$.
	$O \times E = E$.	$E + E = E$.

Page 54

Answers for Exercise B — The truth of both statements is proved in Exercise A above.

Exercise C — Yes; $\underline{E(E + O)}$ and $\underline{(E \times E) + (E \times O)}$ are the same member of M.

Page 54 (cont.)

Answers for Exercise D — a was replaced by E, b by E, and c by 0.

Exercise E — Reference to Exercise A above will show that each of the sentences expresses a true statement.

Exercise F

$E(0 + E) = (E \times 0) + (E \times E)$.

$0(0 + E) = (0 \times 0) + (0 \times E)$.

$E(E + E) = (E \times E) + (E \times E)$.

$0(0 + 0) = (0 \times 0) + (0 \times 0)$.

Exercise G

$E(0 + E) = E \times 0$. $\quad (E \times 0) + (0 \times E) = E + E$.

$E \times 0 = E$. $\quad E + E = E$.

Since $E(0 + E)$ and $(E \times 0) + (0 \times E)$ are the same member of M, the first sentence expresses a true statement.

The truth of the statements expressed by the other three sentences in Exercise F can be checked similarly.

Exercise H — Since each of the sentences in Display 7 and in the answer to Exercise F expresses a true statement, we know that multiplication distributes over addition in set M.

Exercise I — Set M is a number system relative to addition and multiplication because it satisfies the definition of a number system given in Lesson 10.

This definition requires closure under two operations, the properties of commutativity and associativity for each operation, and distribution of one of the operations over the other.

Exercise J — Any objects that make up a set that is a number system relative to two operations are numbers.

Exercise M — {3, 1, 4, 2, 0}

Exercise Q — Yes

A remainder, in a division, is never as great as the divisor.

Exercise R — No; it contains exactly 5 members.

Page 54 (cont.)
Answers for Exercise A — The sum of 4 and 3 is 2.

Page 55
Answers for Exercise C — Divide 7 by 5, and note the remainder. Since the remainder is 2, $4 + 3 = 2$ is a true statement under the operation defined.

Exercise D — If 5 is divided by 5, the remainder is 0.

If 6 is divided by 5, the remainder is 1.

If 8 is divided by 5, the remainder is 3.

Exercise E — 0

Exercise F — 0 and 0, because 0 is under 0 and to the right of 0 in the table

1 and 4, because 0 is under 1 and to the right of 4 in the table

2 and 3, because 0 is under 2 and to the right of 3 in the table

3 and 2, because 0 is under 3 and to the right of 2 in the table

Exercise G — The table in Display 11 names all possible sums onto which two members of R are mapped by the operation of addition as defined. Since each of these sums is a member of R, set R is closed under addition.

Exercise H — The product of 2 and 3 is 1.

Exercise I — Find the product of 2 and 3 by ordinary multiplication. It is 6. Divide 6 by 5. Since the remainder is 1, $2 \times 3 = 1$ is a true statement for the operation multiplication as defined for R.

Exercise J — If 8 is divided by 5 the remainder is 3, so sentence B expresses a true statement.

If 12 is divided by 5 the remainder is 2, so sentence C expresses a true statement.

Exercise K — 3

By ordinary multiplication, the product of 1 and 3 is 3. If 3 is divided by 5, the quotient is 0 and the remainder is 3

Page 55
Answers for Exercise K (cont.)

So 3 is the product of 1 and 3 under the new operation.

Page 56
Answers for Exercise L

×	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

Exercise M — 2 and 4, 3 and 1, 4 and 2

Exercise O — The table names all possible products onto which two members of R are mapped by the operation of multiplication as defined. Since each of these products is a member of R, set R is closed under multiplication.

Exercise Q — 0 0

Exercise R — Yes, because $\underline{4 + 1}$ and $\underline{1 + 4}$ are the same member of R

Exercise S — Sentence B expresses a true statement because $\underline{2 + 4}$ and $\underline{4 + 2}$ are the same member of R.

Sentence C expresses a true statement because $\underline{3 + 4}$ and $\underline{4 + 3}$ are the same member of R.

Exercise T — No; there are twenty-five such statements in all.

Exercise U — No; all twenty-five possible statements are true.

Exercise V — Addition is commutative in set R because $\underline{a} + \underline{b} = \underline{b} + \underline{a}$ yields true statements for all permissible replacements of the variables.

Exercise X — The sum $\underline{2 + 1}$ is found in the table under 2 and to the right of 1.

Page 56 (cont.)

Answers for Exercise Y — The same numeral appears in two positions that are symmetric in respect to the diagonal.

Page 57

Answers for Exercise B — 1 in each case

Exercise C — Sentence A expresses a true statement because $\underline{3 \times 2}$ and $\underline{2 \times 3}$ are the same member of R.

Exercise D — Sentence B expresses a true statement because $\underline{4 \times 1}$ and $\underline{1 \times 4}$ are the same member of R.

Sentence C expresses a true statement because $\underline{3 \times 4}$ and $\underline{4 \times 3}$ are the same member of R.

Exercise E — No; the table in Exercise L shows that this is not possible.

Exercise F — Multiplication is commutative in set R because $\underline{ab} = \underline{ba}$ yields a true statement for each replacement of the variables by members of R.

Exercise G — The numeral representing the product of 1 and 3 occupies a position below the main diagonal, and the numeral representing the product of 3 and 1 occupies a position symmetric to it above the main diagonal. The same observation holds for each such pair of products whose numerals appear in the table. In every case the two numerals are the same.

Exercise I — Yes, because 1 is the numeral in the addition table below 4 and to the right of 2.

4

Exercise J — The development in Exercise I, page 57, shows that $(4 + 2) + 3 = 4$ is a true statement.

Exercise K — 0 4

Exercise L — The development in Exercise K, page 57, shows that $4 + (2 + 3) = 4$ is a true statement.

Exercise M — Yes; because it is the same as $4 = 4$

Page 57 (cont.)
Answers for Exercise N $(1 + 3) + 4 = 4 + 4.$ $1 + (3 + 4) = 1 + 2.$

$4 + 4 = 3.$ $1 + 2 = 3.$

Sentence B expresses a true statement because $(1 + 3) + 4$ and $1 + (3 + 4)$ are the same member of R.

Exercise O 125

There are 25 possible statements that can be obtained from $\underline{a} + \underline{b} = \underline{b} + \underline{a}$. For each of these, the variable $\underline{c}$ can be replaced in 5 different ways.

Exercise Q No, this is not possible.

Exercise R Addition is associative in set R because for any replacements from R for the variables in $(\underline{a} + \underline{b}) + \underline{c} = \underline{a} + (\underline{b} + \underline{c})$, a true statement is obtained.

Exercise S C $(4 \cdot 2)\,3 = 3 \cdot 3.$ $4\,(2 \cdot 3) = 4 \cdot 1.$

$3 \cdot 3 = 4.$ $4 \cdot 1 = 4.$

Sentence C expresses a true statement because $(4 \cdot 2)3$ and $4(2 \cdot 3)$ are the same member of R.

D $(1 \cdot 3)4 = 3\,(4).$ $1(3 \cdot 4) = 1(2).$

$3(4) = 2.$ $1(2) = 2.$

Sentence D expresses a true statement because $(1 \cdot 3)\,4$ and $1(3 \cdot 4)$ are the same member of R.

Exercise T None of the 125 possible sentences expresses a false statement.

Exercise U Multiplication is associative in set R, because a true statement is obtained from the condition $(ab)c = a(bc)$ for each replacement of the variables by members of R.

Page 58
Answers for Exercise V A $2(4 + 3) = 2(2).$ $2(4) + 2(3) = 3 + 1.$

$2(2) = 4.$ $3 + 1 = 4.$

B $3(2 + 1) = 3(3).$ $3(2) + 3(1) = 1 + 3.$

$3(3) = 4.$ $1 + 3 = 4.$

Page 58
Answers for Exercise V (cont.)

C $2(0 + 4) = 2(4)$. $2(0) + 2(4) = 0 + 3$.
$2(4) = 3$. $0 + 3 = 3$.

The above computations show that each sentence in Display 18 expresses a true statement.

Exercise W There are 125 such sentences. Each of them expresses a true statement.

Exercise X Multiplication distributes over addition in set R because a true statement is obtained from the condition $a(b + c) = ab + ac$ for each replacement of the variables by members of R.

Exercise Y Set R is a number system relative to the new operations of addition and multiplication because it is closed under both of these operations; each operation is commutative and associative, and multiplication distributes over addition.

Exercise B Under 3 in the addition table and to the right of 4, we find the numeral 2. So $3 + 4 = 2$ is a true statement. Since {3} is the solution set of the condition $2 = x + 4$ in the universe R, (2,4) can be mapped onto 3 by subtraction.

Exercise C (2,3) can be mapped onto 4.

(1,4) can be mapped onto 2.

(4,3) can be mapped onto 1.

Exercise D No. This can be proved by testing all possible examples.

Exercise E Set R is closed under subtraction. Each member of $R \times R$ can be mapped onto a member of R.

Page 59
Answers for Exercise G Under 3 and to the right of 4 in the multiplication table, we find the numeral 2. So $3 \cdot 4 = 2$ is a true statement. Since {3} is the solution set of the condition $2 = x \cdot 4$ in the universe R, (2,4) can be mapped onto 3 by division.

Page 59 (cont.)
Answers for Exercise H

(3,1) can be mapped onto 3.

(4,3) can be mapped onto 3.

(2,3) can be mapped onto 4.

Exercise I

Set R is not closed under division; an ordered pair whose second component is 0 cannot be mapped by division onto a member of R.

Exercise J

Two

One counterexample is sufficient to prove that a set S is not a number system relative to two given operations.

From the $*$ table we find

$$\underline{c} * \underline{d} = \underline{d}.$$
$$\underline{d} * \underline{c} = \underline{c}.$$

Therefore, $\underline{c} * \underline{d} = \underline{d} * \underline{c}$ is not a true statement, and the operation $*$ is not commutative.

Pages 59-60
Answers for "On Your Own" exercises

1 Yes, because it contains all natural numbers that are less than 3

2 Set F is closed under both addition and multiplication because the tables contain all possible sums and products of two members of F; and all numbers that are named in the tables are members of F.

3 We know immediately from the symmetry with respect to the diagonal from left to right in the tables that both operations are commutative.

4 To prove that both addition and multiplication are associative, we must test 27 sentences of the form $(\underline{a} + \underline{b}) + \underline{c} = \underline{a} + (\underline{b} + \underline{c})$ and 27 sentences of the form $(\underline{ab})\underline{c} = \underline{a}(\underline{bc})$. Since all of these express true statements, we know that both addition and multiplication are associative in set F.

5 To prove that multiplication is distributive over addition in set F, we test 27 sentences of the form $\underline{a}(\underline{b} + \underline{c}) = \underline{ab} + \underline{ac}$. Since all 27 sentences express true statements, multiplication is distributive over addition in set F.

Answers for "On Your Own" exercises

6 Set F is a number system relative to the operations of addition and multiplication since it is closed under both operations, each operation is commutative and associative, and multiplication distributes over addition.

7 Yes, because it contains all natural numbers that are less than 7

8 $G = \{0, 1, 2, 3, 4, 5, 6\}$.

+	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	1	2	3	4	5	6	0
2	2	3	4	5	6	0	1
3	3	4	5	6	0	1	2
4	4	5	6	0	1	2	3
5	5	6	0	1	2	3	4
6	6	0	1	2	3	4	5

9 Set G is closed under addition. We know this since the table above includes all possible sums, and each number represented in the table is a member of G.

10 The symmetry of placement of repeating numerals with respect to the main diagonal in the table shows that addition is commutative in set G.

To prove that addition is associative in set G, we test 343 sentences of the form $(a + b) + c = a + (b + c)$. Since each sentence expresses a true statement, addition is associative in G.

11 $G = \{0, 1, 2, 3, 4, 5, 6\}$.

×	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6
2	0	2	4	6	1	3	5
3	0	3	6	2	5	1	4
4	0	4	1	5	2	6	3
5	0	5	3	1	6	4	2
6	0	6	5	4	3	2	1

Answers for "On Your Own" exercises

12 Set G is closed under multiplication. We know this since the table in Exercise 11 includes all possible products, and each number represented in the table is a member of G.

13 The symmetry of placement of repeating numerals with respect to the main diagonal in the table shows that multiplication is commutative in set G.

To prove that multiplication is associative in set G, it is necessary to test 343 sentences of the form $(\underline{a}\underline{b})\underline{c} = \underline{a}(\underline{b}\underline{c})$. Testing would prove each of these sentences expresses a true statement, so multiplication is associative in G.

14 To prove that multiplication distributes over addition in set G, it is necessary to test 343 sentences of the form $\underline{a}(\underline{b} + \underline{c}) = \underline{a}\underline{b} + \underline{a}\underline{c}$. Testing would verify that each sentence expresses a true statement. So multiplication distributes over addition in G.

15 Set G is a number system relative to the operations of addition and multiplication since it is closed under both operations, each operation is commutative and associative, and multiplication distributes over addition.

16	{2}	18	{2}	20	{1}	22	{2}
17	{1}	19	{1}	21	{2}	23	{0}

LESSON 12

Pages 60-67 Notes

Sometimes an operation defined on a set has properties other than closure, commutativity and associativity.

An operation relative to a set of objects may have an identity element. Suppose that a binary operation '$*$' has been defined on set S. If there is an element $\underline{e}$ in S such that a true statement is obtained from $\underline{e} * \underline{a} = \underline{a}$ for each replacement of $\underline{a}$ from S, then $\underline{e}$ is the identity element for operation '$*$' relative to set S.

To determine whether the operation of addition relative to the natural numbers has an identity element, we must find a natural number $\underline{e}$ such that a true statement is obtained from $\underline{e} + \underline{a} = \underline{a}$ for each replacement of $\underline{a}$ from N.

We will prove that 0 is the identity element for addition of natural numbers.

0 is associated with{ }. Let $\underline{a}$ be the number associated with set A. By definition, the number associated with $\{\} \cup A$ is $\underline{0+a}$. But the union of the empty set and any set A is set A. So the number associated with A is also $\underline{0+a}$. Since just one natural number is associated with a given set, $\underline{0+a}$ and $\underline{a}$ must be the same number. Hence a true statement is obtained from $0 + \underline{a} = \underline{a}$ for each replacement of $\underline{a}$ from N. Hence, 0 is the identity element for addition of natural numbers.

To determine whether the operation of multiplication relative to the natural numbers has an identity element, we must find a natural number $\underline{e}$ such that a true statement is obtained from $\underline{e} \times \underline{a} = \underline{a}$ for each replacement of $\underline{a}$ from N. We will prove that 1 is the identity element for multiplication of natural numbers. 1 is associated with any set that has just one member. Let $1 = \underline{n}(B)$ and $\underline{a} = \underline{n}(A)$. By definition, the number associated with B×A is $\underline{1 \times a}$. Since B×A and set A are equivalent sets, B×A and set A have the same number of members. Therefore, $\underline{1 \times a}$ is the same number as $\underline{a}$, and a true statement is obtained from $1 \times \underline{a} = \underline{a}$ for each replacement of $\underline{a}$ from N. 1 is the identity element for multiplication of natural numbers.

To determine whether the operation of subtraction relative to the natural numbers has an identity element, we must find a natural number $\underline{e}$ such that a true statement is obtained from $\underline{e} - \underline{a} = \underline{a}$ for each replacement of $\underline{a}$ from N. Since there is no natural number that satisfies this requirement, there is no identity element for subtraction of natural numbers. Likewise, there is no natural number $\underline{e}$ such that a true statement is obtained from $\underline{e} \div \underline{a} = \underline{a}$ for each replacement of $\underline{a}$ from N. So there is no identity element for division of natural numbers.

The identity element for addition, which is zero for the natural numbers, is related to another property of the natural numbers. This property is the zero property of multiplication. To prove this property, we must show that a true statement is obtained from $0 \times \underline{a} = 0$ for each

replacement of $\underline{a}$ from N.

0 is associated with $\{\ \}$. Let $\underline{a} = \underline{n}(A)$. By definition, $\underline{0 \times a}$ is associated with $\{\ \} \times A$. However, $\{\ \} \times A$ is the set of ordered pairs whose first components are members of the empty set and whose second components are members of A. Since there are no ordered pairs whose first components are members of the empty set, $\{\ \} \times A$ is the empty set. Therefore, 0 is associated with $\{\ \} \times A$. Since just one number is associated with a given set, $\underline{0 \times a}$ and 0 are the same number. Since a true statement is obtained from $0 \times \underline{a} = 0$ for each replacement of the variable from N, $0 \times \underline{a} = 0$ is a property of the natural numbers.

Page 60
Answers for Exercise A 0 4

Exercise B $E \cup F = \{5, 6, 7, 8\}$.

Yes; the sum of two natural numbers is the number associated with the union of sets with which those numbers are separately associated.

Exercise D Yes; the sum $\underline{0+4}$ is associated with $E \cup F$, and the number 4 is associated with F. Since $E \cup F$ and F are the same set, $\underline{0+4}$ is the same number as 4.

Exercise E 5 3 0

Exercise F {1, 3, 5, 7, 9}

Yes, because the sum of two natural numbers is the number associated with the union of sets with which those numbers are separately associated

Exercise G {10, 11, 12}

The sum $\underline{0+3}$; because the number associated with $E \cup D$ is, by definition, the sum $\underline{n}(E) + \underline{n}(D)$

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Answers for Exercise H Yes; see answer to Exercise F above.

Yes; because the sum $0+5$ is associated with the set $E \cup C$, and the number 5 is associated with the set C;

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Answers for Exercise H (cont.) since $E \cup C$ is the same set as C, $\underline{0+5}$ is the same number as 5.

Exercise I Yes; see answer to Exercise G above.

Yes; because the sum $\underline{0+3}$ is associated with the set $E \cup D$, and the number 3 is associated with the set D and since $E \cup D$ is the same set as D, $\underline{0+3}$ is the same number as 3.

Exercise J Yes, because the members of A and the members of the empty set are the members of A

Exercise K $\{\} \cup A$

Yes, because $\{\} \cup A$ is the same set as A

Exercise L No member besides 0 is an identity element for addition because $B \cup A = A$ is a true statement for disjoint sets only if B is the empty set.

Exercise M A: $\underline{a}$ has been replaced by 8.

B: $\underline{a}$ has been replaced by 99.

C: $\underline{a}$ has been replaced by 0.

Exercise N Yes, because $\underline{a} + 0 = \underline{a}$ becomes a true statement for each replacement of $\underline{a}$ by a member of N

Exercise O The number itself

The identity-element property for addition of natural numbers

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Answers for Exercise P Yes; because for each replacement of $\underline{a}$, a true statement is obtained from $\underline{a} - \underline{a} = 0$.

Exercise Q Yes; because for each replacement of $\underline{a}$, a true statement is obtained from $\underline{a} - 0 = \underline{a}$.

Exercise R No; the statement is true only if $\underline{a}$ is replaced by 0.

Exercise S No; suppose that the natural number 8 is such that a true statement is obtained from $8 - \underline{a} = \underline{a}$ for each replacement of $\underline{a}$ from N. This means that a true statement is obtained from $8 = \underline{a} + \underline{a}$ for each replacement

Page 62

Answers for Exercise S (cont.) of a from N. But a+a becomes a different number for each replacement of a from N. So the number 8 cannot always be the number that a+a becomes. Hence a true statement is not always obtained from $8 = a + a$, and therefore, a true statement is not always obtained from $8 - a = a$. So the natural number 8 cannot be the identity element for subtraction. A similar proof would apply to any other natural number we choose.

Exercise A 1 3

Exercise B $M \times N = \{(6,7),(6,14),(6,21)\}$.

Yes, because the product of any two natural numbers is associated with the Cartesian set of the two sets with which the two numbers are separately associated

Exercise C Yes, because the two sets can be put in one-to-one correspondence

Exercise D Yes, because the set associated with $1 \cdot 3$ and the set associated with 3 are equivalent sets

Exercise E 1 4 6

Exercise F $J \times K = \{(2,5),(2,10),(2,15),(2,20)\}$.

Yes, because the product of any two natural numbers is associated with the Cartesian set of the two sets with which the two numbers are separately associated

Exercise G $J \times L = \{(2,1), (2,3), (2,5), (2,7), (2,9),(2,11)\}$.

$1 \cdot 6$ is associated with this set, because 1 is associated with set J, 6 is associated with set L, and the product of any two natural numbers is associated with the Cartesian set of the two sets with which the two numbers are separately associated.

Exercise H Yes; because the two sets can be put in one-to-one correspondence

Yes; because the product $1 \cdot 4$ is associated with $J \times K$, the number 4 is associated with K, and K is equivalent to $J \times K$.

Page 62 (cont.)

Answers for Exercise I — The answer is similar to the answer given in Exercise H.

Exercise J — Yes; suppose that 1 is the only member of B and x is a variable for the members of A. Map each x in A onto $(x, 1)$ in B×A. Map each $(x,1)$ in B×A onto x in A. These two mappings form a one-to-one correspondence between A and B×A. So A and B×A are equivalent sets.

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Answers for Exercise K — B×A, or A, since B×A and A, under the condition set up in the problem, are equivalent sets

Yes, since $1 \cdot a$ represents the number of members in B×A, and a represents the number of members in A

Yes, because the number associated with each of two equivalent sets is always the same number

Exercise L — A×B and B are equivalent sets only when the number associated with A is 1. So 1 is the only member of N that is an identity element for multiplication.

Exercise M — A: a has been replaced by 16.

B: a has been replaced by 43.

C: a has been replaced by 1.

Exercise N — Yes; because for each replacement of a, a true statement is obtained from $a \cdot 1 = a$. This follows from the truth of statements obtained from the condition $1 \cdot a = a$ by the commutative property of multiplication.

Exercise P — a, that is the natural number itself

The identity-element property for multiplication of natural numbers

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Answers for Exercise Q — Yes; because it yields true statements for all replacements of a from N

Exercise R — Since $a = 1 \cdot a$ yields a true statement for each replacement of a from N, we know, because of the commutative property of multiplication, that $a = a \cdot 1$

Page 64

Answers for Exercise R (cont.) also yields a true statement for each such replacement. So $a/1 = a$ is a property.

Exercise S No; it does not yield a true statement for all replacements of a from N. So $1/a = a$ is not a property.

Exercise T No; suppose the natural number 9 is such that a true statement is obtained from $9/a = a$ for each replacement of a from N. This means that a true statement is obtained from $9 = a \cdot a$ for each replacement of a from N. But $a \cdot a$ becomes a different number for each replacement of a from N. So the number 9 cannot always be the number that $a \cdot a$ becomes. Hence a true statement is not always obtained from $9 = a \cdot a$, and therefore a true statement is not always obtained from $9/a = a$. So the natural number 9 cannot be the identity element for division. A similar proof would apply to any other natural number we choose.

Exercise A The set of even numbers

The set of odd numbers

Exercise B The set of even numbers contains the identity element for addition because 0 is an even natural number.

The set of even numbers does not contain an identity element for multiplication because 1 is not an even number.

Exercise C The set of odd numbers does not contain the identity element for addition because 0 is not an odd number.

The set of odd numbers does contain the identity element for multiplication because 1 is an odd number.

Exercise D The tables can be used to check this. For example, $E + E = E$ is a true statement because in the addition table, E is under E and to the right of E.

Exercise E Yes; because the condition $x + E = x$ yields a true statement when x is replaced by either member of {E, O}

Exercise F When E is added to O, the sum is O.

When O is added to O, the sum is E.

Page 64 (cont.)

Answers for Exercise G — Because the condition $\underline{x} + 0 = \underline{x}$ does not yield a true statement when $\underline{x}$ is replaced by a member of {E, 0}

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Answers for Exercise H — Sentence F tells you that when you multiply 0 by E you get E.

Sentence G tells you that when you multiply E by 0 you get E.

Sentence H tells you that when you multiply E · E you get E.

Exercise I — Yes

The condition $\underline{x} \cdot 0 = \underline{x}$ yields a true statement when $\underline{x}$ is replaced by either member of {E, 0}.

Exercise J — The condition $\underline{x} \cdot E = \underline{x}$ does not yield a true statement when $\underline{x}$ is replaced by 0.

Exercise A — 0 2

Exercise B — None

Exercise C — Yes; both are the empty set since neither has any members.

Exercise D — 0

Yes; the number associated with set E is 0, and the number associated with set F is 2. By definition, the number associated with E × F is $\underline{0 \times 2}$.

Exercise E — Yes

$\underline{0 \cdot 2}$ is associated with E×F, and 0 is associated with E.

Since E and E×F are equivalent sets, $\underline{0 \cdot 2}$ and 0 are the same number.

Exercise F — { }×A will always be the same as the empty set because { } has no members that can be used as first or second components for form ordered pairs in the Cartesian set. Therefore the Cartesian set, in this case, is the empty set.

Page 65 (cont.)
Answers for Exercise G { }×A

Since { } × A = { } , $\underline{0}\cdot\underline{a}$ is the number associated with { }.

Exercise H A: $\underline{a}$ has been replaced by 17.

B: $\underline{a}$ has been replaced by 98.

C: $\underline{a}$ has been replaced by 0.

Exercise I It is a property because the condition $0 \cdot \underline{a} = 0$ yields a true statement for each replacement of $\underline{a}$ from N.

Page 66
Answers for Exercise J Yes, by the zero property for multiplication of natural numbers

Yes, because the condition $0/\underline{a} = 0$ means to find replacements for $\underline{a}$ that satisfy $0 = 0 \cdot \underline{a}$

Exercise K A: $\underline{a}$ has been replaced by 14.

B: $\underline{a}$ has been replaced by 97.

C: $\underline{a}$ has been replaced by 146.

We know this because of the zero property for multiplication of natural numbers. See Exercise J above.

Exercise L If you divide 0 by any natural number other than 0, you obtain 0. What happens when an attempt is made to divide 0 by 0 is shown on page 66, column 2.

Exercise M {1, 2, 3, ···}

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Answers for "On Your Own" exercises

1	16	9	78
2	126	10	22
3	12	11	0
4	3	12	0
5	0	13	15
6	157	14	1
7	33	15	1
8	72	16	0

Answers for "On Your Own" exercises

17	{1}	21	{0, 1, 2, 3, ···}	25	{7}	29	{0}
18	{5}	22	{1}	26	{0}	30	{0}
19	{0}	23	{1, 2, 3, ···}	27	{1}		
20	{1}	24	{0}	28	{9}		

31 0 is the identity element for addition since all the following are true statements.

$$0 + 0 = 0.$$
$$0 + 1 = 1.$$
$$0 + 2 = 2.$$

32 1 is the identity element for multiplication since all the following are true statements.

$$1 \cdot 0 = 0.$$
$$1 \cdot 1 = 1.$$
$$1 \cdot 2 = 2.$$

LESSON 13

Pages 67-73 Notes

Factorization and primes are topics that receive much attention in a branch of mathematics called Number Theory. They have aroused the curiosity of many famous mathematicians; and included in the theory of prime numbers, there are several unsolved problems. Attention is called to them in the lesson, and they will be summarized in the concluding paragraph of these notes.

To develop skill in factoring natural numbers, the student must learn to see his multiplication tables in reverse. Knowledge that $7 \times 8 = 56$ is first acquired with emphasis on the 56. When the student studies factoring, the emphasis is on the 7 and the 8. Since $7 \times 8 = 56$ is a true statement, 7 and 8 are factors of 56. Numbers other than 7 and 8 are also factors of 56 because all the following are true statements.

$$1 \times 56 = 56.$$
$$2 \times 28 = 56.$$
$$4 \times 14 = 56.$$

From the four statements under consideration, we know that {1, 2, 4, 7, 8, 14, 28, 56} is the set of all factors of 56 in the universe of counting numbers.

To find the factors of a given number, it is necessary sometimes to use several divisions. If the number is large it is helpful to know some ways of deciding what divisors to try. Here are some generalizations about the use of base-ten numerals in finding possible divisors, but numerals written in other bases are not subject to them.

1 If the last digit of the numeral is 0, 2, 4, 6, or 8, the number that it expresses is divisible by 2.

2 If the number expressed by the sum of the digits is divisible by 3, the number itself is divisible by 3.

3 If a numeral ends in 0 or 5, it expresses a number divisible by 5.

If the number to be factored is 546, these generalizations tell us that 2 and 3 are factors of it. Then it is fairly easy to see that 7 divides 91 with no remainder. We can conclude then that some of the factors of 546 are 2, 3, 7, and 13.

2	546
3	273
7	91
	13

The content of pages 68-72 is concerned with generalizations that students can make from their charts for finding the factors of numbers from 1-100. This chart, including the names of natural numbers from 1 through 100, is given below. For each number expressed in the chart, all factors of that number, with the exception of the number itself and 1 are expressed. Since the only factors of a prime number are the number itself and 1, the factors of the prime numbers between 1 and 100 are not expressed.

1	
2	
3	
4	2
5	
6	2, 3
7	
8	2, 4
9	3
10	2, 5

11
~~12~~ 2, 3, 4, 6
13
~~14~~ 2, 7
~~15~~ 3, 5
~~16~~ 2, 4, 8
17
~~18~~ 2, 3, 6, 9
19
~~20~~ 2, 4, 5, 10
~~21~~ 3, 7
~~22~~ 2, 11
23
~~24~~ 2, 3, 4, 6, 8, 12
~~25~~ 5
~~26~~ 2, 13
~~27~~ 3, 9
~~28~~ 2, 4, 7, 14
29
~~30~~ 2, 3, 5, 6, 10, 15
31
~~32~~ 2, 4, 8, 16
~~33~~ 3, 11
~~34~~ 2, 17
~~35~~ 5, 7
~~36~~ 2, 3, 4, 6, 9, 12, 18
37
~~38~~ 2, 19
~~39~~ 3, 13
~~40~~ 2, 4, 5, 8, 10, 20
41
~~42~~ 2, 3, 6, 7, 14, 21
43
~~44~~ 2, 4, 11, 22
~~45~~ 3, 5, 9, 15
~~46~~ 2, 23
47
~~48~~ 2, 3, 4, 6, 8, 12, 16, 24
~~49~~ 7
~~50~~ 2, 5, 10, 25
~~51~~ 3, 17
~~52~~ 2, 4, 13, 26
53
~~54~~ 2, 3, 6, 9, 18, 27
~~55~~ 5, 11
~~56~~ 2, 4, 7, 8, 14, 28
~~57~~ 3, 19
~~58~~ 2, 29
59
~~60~~ 2, 3, 4, 5, 6, 10, 12, 15, 20, 30
61
~~62~~ 2, 31
~~63~~ 3, 7, 9, 21
~~64~~ 2, 4, 8, 16, 32
~~65~~ 5, 13
~~66~~ 2, 3, 6, 11, 22, 33
67
~~68~~ 2, 4, 17, 34
~~69~~ 3, 23
~~70~~ 2, 5, 7, 10, 14, 35
71
~~72~~ 2, 3, 4, 6, 8, 9, 12, 18, 24, 36

73	
~~74~~	2, 37
~~75~~	3, 5, 15, 25
~~76~~	2, 4, 19, 38
~~77~~	7, 11
~~78~~	2, 3, 6, 13, 26, 39
79	
~~80~~	2, 4, 5, 8, 10, 16, 20, 40
~~81~~	3, 9, 27
~~82~~	2, 41
83	
~~84~~	2, 3, 4, 6, 7, 12, 14, 21, 28, 42
~~85~~	5, 17
~~86~~	2, 43
~~87~~	3, 29
~~88~~	2, 4, 8, 11, 22, 44
89	
~~90~~	2, 3, 5, 6, 9, 10, 15, 18, 30, 45
~~91~~	7, 13
~~92~~	2, 4, 23, 46
~~93~~	3, 31
~~94~~	2, 47
~~95~~	5, 19
~~96~~	2, 3, 4, 6, 8, 12, 16, 24, 32, 48
97	
~~98~~	2, 7, 14, 49
~~99~~	3, 9, 11, 33
~~100~~	2, 4, 5, 10, 20, 25, 50

Observe from the preceding table, that each numeral that expresses a number having more than two factors has been crossed out. Numbers having more than two factors are called composite numbers. Names of numbers having exactly two factors are not crossed out. These numbers are prime numbers. To find all composite numbers from 1 through 100, it is necessary only to consider divisors less than 8. Numbers that are divisible by 8, 9, or 10 have at least one factor less than 7, and any number less than 100 that is divisible by a number greater than 10 cannot be expressed as the product of two numbers, each greater than 10.

The able learners may wish to examine the table in more detail than is required in the lesson. The following questions could be considered by these students.

1 Which numbers have just three factors? What other common property do these numbers have?

Answer:

The numbers 4, 9, 25, and 49 each have exactly three factors.

Each natural number having three factors is a square of a prime number.

2 Which numbers in the table have a prime number of factors? What general statement describes all of these numbers?

Answer:

The numbers 2, 3, 4, 5, 7, 9, 11, 16, 25, 49, and 81 are some of the numbers having a prime number of factors. The number of factors of a natural number n $(n > 0)$ is a prime number if and only if $n = q^{P-1}$ where both p and q are primes.

3 What is the sum of the factors of a prime number?

Answer:

The sum of the factors of a prime number is the sum of the prime and the number 1.

Additional generalizations such as the number of factors of any natural number, the sum of the factors of any number, and the number of primes less than or equal to a given natural number may also be made from the preceding table. A discussion of these topics can be found in books dealing with number theory.

One of the major questions about primes was solved by Euclid. He proved that the number of primes is infinite, or, in other words, there is no greatest prime. Euclid's proof is given below.

Assume that there is a largest prime. Represent it by p.

Form the product P of all primes $\leq p$, and add 1.

$$P + 1 = 2 \cdot 3 \cdot 5 \cdot 7 \cdots \ . p + 1.$$

Either P + 1 is prime or not prime.

If P + 1 is prime, then p is not the greatest prime, because P + 1 is greater than p.

If P + 1 is not prime it must be the product of primes each of which is greater than p. This is true because when P + 1 is divided by any prime $\leq p$, there is a remainder of 1. So there must be at least one prime greater than p .

Therefore, there is no greatest prime.

There are still many unanswered questions in the theory of prime numbers. Some of these are listed below.

1 Is there a systematic way of forming prime numbers, or of testing to see whether or not a given number is prime?

2 Is there an infinite number of twin primes?

3 Is every even number the sum of two primes? A German mathematician conjectured that every even number except 2 is the sum of two primes. This is true for every even number he examined, but he could not prove that it is true for all even numbers.

4 Perfect numbers were defined by the Greeks to be numbers which are equal to the sum of their proper divisors. A proper divisor of a number is any divisor except one number itself. 6, 28, and 496 are perfect numbers in this sense. All perfect numbers that have been found are even numbers, but no one has proved that an odd perfect number does not exist.

Page 67
Answers for Exercise A

A: $\underline{a}$ was replaced by 3, $\underline{b}$ by 2, and $\underline{c}$ by 6.

B: $\underline{a}$ was replaced by 6, $\underline{b}$ by 1, and $\underline{c}$ by 6.

Page 68
Answers for Exercise B

Since $3 \times 2 = 6$ is a true statement, 2 is a factor of 6.

Since $6 \times 1 = 6$ is a true statement, 6 is a factor of 6.

Exercise C

Yes, because (1,6), (6,1), (2,3), and (3,2) are the only ordered pairs of counting numbers that multiplication maps onto 6

Exercise D

{1, 2, 4, 8}

{1, 2, 3, 4, 6, 12}

{1, 13}

Exercise E

Yes, because $1 \cdot \underline{a} = \underline{a}$ is a true statement for each replacement of $\underline{a}$ by a member of {1, 2, 3, ···}

Page 68 (cont.)

Answers for Exercise F — Yes, because $2 \cdot 1 = 2$ is a true statement

Exercise J — {2, 4, 6, 8, 10}; yes

Exercise K — No; {1, 3, 5, 7, 9, 11}

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Answers for Exercise O — Both 2 and 3 are factors of 6.

12, 18, 24, 30

Exercise P — 1 and 4 are factors of 4 because $1 \times 4 = 4$ is a true statement.

{1, 2, 4}

Exercise Q — {1, 2, 3, 6}

{1, 2, 4, 8}

{1, 3, 9}

{1, 2, 5, 10}

{1, 2, 3, 4, 6, 12}

Exercise R — The identity-element property of multiplication shows that every member of {1, 2, 3,···} must have 1 as a factor and itself as a factor.

Exercise A — This table is shown on pages 138-140.

Page 69

Answers for Exercise C — {6, 12, 18, 24, 30}

6

Exercise D — The numeral 4 is crossed out because the number 4 has a factor of 2.

Yes, because each number that has 4 as a factor also has 2 as a factor

Page 70

Answers for Exercise F — Yes, because any number having 6 as a factor also has 2 and 3 as factors

Exercise G — 13 should not be crossed out because it has only two factors. They are 13 and 1.

Page 70 (cont.)

Answers for Exercise H

Every number, except 1, has at least two factors. They are itself and 1. All numerals that are crossed out are names of numbers having at least three factors.

Exercise I

4 is a composite number because it has 3 factors. They are 1, 2, and 4.

{4, 6, 8, 9, 10, 12, 14, 15, 16, 18}

Exercise J

{20, 21, 22, 24, 25, 26, 27, 28, 30}

No, because it has some members that do not contain 2 as a factor

Exercise K

Yes; each numeral crossed out has at least 3 factors. For example, 38 is crossed out because it contains the factor 2. So it contains the factors 1, 2, 19, and 38.

Exercise L

No, because each numeral not crossed out is the name of a number that has no factors other than itself and 1

Exercise M

{2, 3, 5, 7}

1 is not a member of the set, because it has only 1 factor. The set of factors of 1 is {1}.

Exercise N

1 is not a prime number because it does not have two different factors.

2 is a prime number because it has exactly two factors. They are itself and 1.

3 is a prime number because it has exactly two factors. They are itself and 1.

Exercise O

{2, 4, 6, 8, 10}

2 is a prime number.

Page 71

Answers for Exercise P

12 is not a prime number. Its factors are 1, 2, 3, 4, 6, and 12. 14 is not a prime number. Its factors are 1, 2, 7, and 14.

No even number greater than 2 is a prime number because each such number has at least 3 factors. They are 1, itself, and 2.

Page 71 (cont.)

Answers for Exercise Q

Each member of set S has exactly two factors, so each member is a prime number.

S does not contain all prime numbers, for example 23.

Exercise S

{1, 5, 25}

{1, 7, 49}

Three

The three factors are 1, the prime number, and the square of the prime number.

Exercise T

No

One

All the members of {1, 2, 3, ···} have at least two factors except the number 1. The set of its factors is {1}.

Exercise U

Yes

The number $1 \times 2 \times 3 \times 4 \times 5 \times \cdots$ is a member of {1, 2, 3,···}, and its set of factors is an infinite set.

Exercise V

No, because the number 1 is a counting number and 1 is neither a prime nor a composite number

Exercise A

The chart below contains the names of all prime numbers from 61 through 100.

61 through 70	61		67	
71 through 80	71	73		79
81 through 90		83		89
91 through 100			97	

Exercise B

A deduction from answer to Exercise A suggests that the pattern for number of primes between the intervals is erratic. The student who investigates this will find that

{101, 102, 103,···, 110} includes 4 primes. They are 101, 103, 107, and 109.

{111, 112, 113,···, 120} includes one prime. It is 113.

Page 71 (cont.)

Answers for Exercise C — Opinions expressed may differ. Several centuries B.C., Euclid proved that there is no greatest prime. Euclid's proof is given in the notes for this lesson.

Exercise D — 3 and 5

Exercise E — 5 and 7, 11 and 13, 17 and 19, 41 and 43

Exercise F — 71 and 73

Exercise G — 8 is expressed as the sum of the prime numbers 5 and 3.
10 is expressed as the sum of the prime numbers 3 and 7.
12 is expressed as the sum of the prime numbers 5 and 7.
14 is expressed as the sum of the prime numbers 7 and 7.

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Answers for Exercise H — One way to express each of the numbers as the sum of two prime numbers is given below. For all but 4 and 6 there are other possibilities.

$4 = 2 + 2.$
$6 = 3 + 3.$
$16 = 5 + 11.$
$20 = 7 + 13.$
$26 = 13 + 13.$
$28 = 5 + 23.$
$34 = 17 + 17.$
$36 = 7 + 29.$

Exercise I — 5 5 and 7

Exercise J — 17, 19, 23, 29, 31
31, 37, 41, 43, 47, 53, 59

Pages 72-73

Answers for "On Your Own" exercises

	Number	Set of Factors
1	27	{1, 3, 9, 27}
2	3	{1, 3}

Answers for "On Your Own" exercises

3	29	{1, 29}
4	32	{1, 2, 4, 8, 16, 32}
5	57	{1, 3, 19, 57}
6	120	{1, 2, 3, 4, 5, 6, 8, 10, 12, 15, 20, 24, 30, 40, 60, 120}
7	167	{1, 167}
8	227	{1, 227}

9 3, 29, 167, and 227 are prime numbers.
27, 32, 57, and 120 are composite numbers.

10 19

11 2

12 3, 5, and 7 are triple primes.

13 K = {1, 2, 4, 8, 16}.
L = {1, 2, 4}.
L is a subset of K because each member of L is also a member of K.

14 One
The number two has just two factors.

	Base-ten numeral	Even or odd
15	4	even
16	7	odd
17	18	even
18	5	odd

19

Ones' digit	Example
0	30
4	24
6	56
8	18

Answers for "On Your Own" exercises

20

Ones' digit	Base-three numeral	Base-ten numeral
0	20	6
1	11	4
2	22	8

The above chart shows that the symbols 0, 1, and 2 can be the ones' digit of a base-three numeral for an even number. But a base-three numeral ending with 0, 1, or 2 is not necessarily the name of an even number. For example, 10_{three}, 21_{three}, and 12_{three} are not even numbers.

21 Even 23 Even 25 Odd 27 Odd

22 Even 24 Even 26 Odd 28 Even

29

Number	Factors	Number of Factors	Sum of Factors
1	1	1	1
2	1, 2	2	3
3	1, 3	2	4
4	1, 2, 4	3	7
5	1, 5	2	6
6	1, 2, 3, 6	4	12
7	1, 7	2	8
8	1, 2, 4, 8	4	15
9	1, 3, 9	3	13
10	1, 2, 5, 10	4	18
11	1, 11	2	12
12	1, 2, 3, 4, 6, 12	6	28
13	1, 13	2	14
14	1, 2, 7, 14	4	24
15	1, 3, 5, 15	4	24
16	1, 2, 4, 8, 16	5	31
17	1, 17	2	18
18	1, 2, 3, 6, 9, 18	6	39

Answers for "On Your Own" exercises

19	1, 19	2	20
20	1, 2, 4, 5, 10, 20	6	42
21	1, 3, 7, 21	4	32
22	1, 2, 11, 22	4	36
23	1, 23	2	24
24	1, 2, 3, 4, 6, 8, 12, 24	8	60
25	1, 5, 25	3	31
26	1, 2, 13, 26	4	42
27	1, 3, 9, 27	4	40
28	1, 2, 4, 7, 14, 28	6	56
29	1, 29	2	30
30	1, 2, 3, 5, 6, 10, 15, 30	8	72

30 2, 3, 5, 7, 11, 13, 17, 19, 23, 29

31 4, 9, 25

32 6 and 28

33 {6, 28}

Note: The next smallest perfect number is 496.

34 Since $12 = 2 \times 6$ and $24 = 2 \times 12$ are true statements, 2 is a factor of each component of (12, 24).

1, 3, 4, 6, 12

35 1, 7

36 1, 2

37 2, 25 1, 4, 5, 10, 20, 50, and 100 are also correct answers.

38 1, 2

39 1, 3

40 2, 13 1 and 26 are also correct answers.

41 Condition a

Answers for "On Your Own" exercises

42 2, 3, and 7

43 89

44 2 and 7

45 2, 11, and 13

46 2 and 5

47 2, 3, and 5

48 2, 3, 5, and 7

49 2, 3, 5, and 11

50 30

51 1

52 10

53 1

LESSON 14

Pages 74-77 Notes

Any composite number is a product of prime numbers. For example, $20 = 2 \times 2 \times 5$, and $33 = 3 \times 11$ are true statements. When a composite number has been factored into primes, we say that we have found the complete factorization of that number. Remember that 1 is not a prime number and does not occur in the complete factorization of a number. For this reason, a prime number is of complete factorization of itself. For example, 7 is the complete factorization of 7.

The complete factorization of a composite number can sometimes be expressed in more than one way because of power notation. The complete factorization of 20 can be expressed as $2 \times 2 \times 5$, or $2^2 \times 5$, The following sentences show the advantage of using power notation. $256 = 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2$. $256 = 2^8$.

Note that the complete factorization of a number is different from

the set of all factors of a number. The complete factorization of 30 is $5 \times 2 \times 3$, but $\{1, 2, 3, 5, 6, 10, 15, 30\}$ is the set of all factors of 30.

To find the complete factorization of a number, we use the operation of division as many times as necessary. The first divisor can be any prime factor of the number. Consider 105. The prime number 5 is a divisor, since $105 = 5 \times 21$ is a true statement. Now 3 is a divisor of 21 and $3 \times 7 \times 5$ is the complete factorization of 105.

Only one complete factorization of a number into primes is possible, except for the order in which the primes occur in the product. This property, the unique factorization property of counting numbers, is of such importance to mathematicians that it is called the fundamental theorem of arithmetic.

Sometimes in a group of numbers, there is a factor or several factors, other than 1, common to all the numbers. Since $21 = 3 \times 7$, and $56 = 8 \times 7$ are true statements, the number 7 is a common factor of 21 and 56. If there is more than one common factor, the greatest of them is called the greatest common factor.

There are two ways to find a greatest common factor. One is to tabulate the set of all factors for each number concerned, and then find the greatest number that is a member of both sets. Sometimes this is very difficult. The alternative is to find the complete factorization of each number, and then find the product of common prime factors using the greatest power of a common prime factor that occurs in both numbers. For example:

$$162 = 2 \cdot 3^4$$
$$378 = 2 \cdot 3^3 \cdot 7.$$

This identifies $2 \cdot 3^3$, or 54, as the greatest common factor of 162 and 378.

When the only common factor is 1, the numbers are said to be relatively prime.

Page 74

Answers for Exercise A

A prime number has exactly two factors. Since 4 has three factors, 1, 2, and 4, it is not a prime number. Since 6 has four factors, 1, 2, 3, and 6, it is not a prime number.

Page 74

Answers for Exercise A (cont.) The prime numbers 2 and 2 can be multiplied to obtain 4.

The prime numbers 2 and 3 can be multiplied to obtain 6.

Exercise B 2 is a prime number. Its factors are 1 and 2.

3 is a prime number. Its factors are 1 and 3.

Therefore, $2 \times 2 \times 2 \times 3$ is an expression of 24 as a product of prime numbers.

Exercise C Yes, because $2^3 = 2 \times 2 \times 2$ is a true statement

Exercise D 20

20 was replaced by 2×10. This can be done because $20 = 2 \times 10$ is a true statement.

Exercise E 10

10 was replaced by 2×5. This can be done because $10 = 2 \times 5$ is a true statement.

Exercise F Yes; 2 and 5 are prime numbers.

$2^2 \times 5^2$ is the same number as $2 \times 2 \times 5 \times 5$ since $2^2 = 2 \times 2$, and $5^2 = 5 \times 5$ are true statements.

Exercise G 10 is not a prime number. Its factors are 1, 2, 5, and 10.

35 is not a prime number. Its factors are 1, 5, 7, and 35.

Exercise H By replacing 10 by 2×5

35 5 and 7

Exercise I Yes; 2, 5, and 7 are prime numbers.

$2 \times 5^2 \times 7$

Page 75

Answers for Exercise J 9, 70

Exercise K Yes, because 2 and 3 are prime numbers

Page 75 (cont.)

Answers for Exercise L

$12 = 2^2 \times 3$.

$16 = 2^4$.

$120 = 2^3 \times 3 \times 5$.

$144 = 2^4 \times 3^2$.

Exercise M — 144 is divisible by 36 since the remainder is 0 when 144 is divided by 36.

Exercise N — No, because when 52 is divided by 14, the remainder is not 0.

Exercise O

$28 = 2^2 \times 7$.

$32 = 2^5$.

$45 = 3^2 \times 5$.

$75 = 5^2 \times 3$.

$98 = 7^2 \times 2$.

$121 = 11^2$.

Exercise P — No, because 9 is not a prime number

Exercise Q — The simplification of each of the expressions is 24.

Exercise R — Yes, because 2 and 3 are prime numbers

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Answers for Exercise S — Three different ways. The product 2×3 is a complete factorization of 6. The number 1 is not a prime number, so it is not included in a complete factorization of 6.

Exercise T — An unlimited number because we can use 1 as a factor of 6 an unlimited number of times

Exercise U — No, because there would be more than one complete factorization for each of the counting numbers

Exercise V

$3^2 \times 2$ is the complete factorization of 18.

$3^2 \times 5$ is the complete factorization of 45.

Exercise W

2 is a factor of 18, so 18 is an even number.

2 is not a factor of 45, so 45 is an odd number.

Exercise A — Set A Set B

Page 76 (cont.)

Answers for Exercise B

Since 2 is a member of set A and also of set B, 2 is a factor of 8 and also a factor of 20.

Exercise C

2 is a common factor of 8 and 20 because it is a factor of both 8 and 20.

4 is a common factor of 8 and 20 because it is a factor of both 8 and 20.

Exercise D

{1, 2, 4}

Yes, because each of its members is a member of set A in Display 6

Yes, because each of its members is a member of set B in Display 6

Exercise E

{1, 2, 3, 4, 6, 12}

{1, 2, 3, 5, 6, 10, 15, 30}

{1, 2, 3, 6}

Exercise F

{1} {1} {1}

Exercise G

1 is a factor of every prime number, and every prime number has no other factors except itself and 1.

Exercise H

{1, 5}

{1, 3}

Exercise I

The set of factors of 12 is {1, 2, 3, 4, 6, 12}.
The set of factors of 16 is {1, 2, 4, 8, 16}.

Since 1, 2, and 4 are the only members common to both sets, {1, 2, 4} is the set of common factors of 12 and 16.

The greatest member of {1, 2, 4} is 4.

Exercise J

Yes

{1, 2, 7, 14} is the set of factors of 14.
{1, 2, 7, 14, 28} is the set of factors of 28.
So {1, 2, 7, 14} is the set of common factors of 14 and 28.
The greatest common factor of 14 and 28 is 14.

Exercise K

{1, 2, 13, 26}

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Answers for Exercise K (cont.) No, because 26 has no factor greater than itself

Exercise L There is a set of factors of 100, and there is also a set of factors of 210. The numbers that are the common factors of 100 and 210 are the factors that are members of both sets. Tabulate both sets, then choose the common members.

The greatest common factor of 100 and 210 is the greatest number that is a member of the set of common factors.

Exercise M {1, 2, 5, 10} 10

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Answers for Exercise O 2 and 5

10

Yes. See Exercise M above.

Exercise P $300 = 2^2 \times 3 \times 5^2$.

$550 = 2 \times 5^2 \times 11$.

Exercise Q 2 and 5

Exercise R 10

No; there is a common factor greater than 10.

Yes; the highest common factor of 300 and 550 is 50, and $2 \times 5^2 = 50$ is a true statement.

Exercise B In the factorization of 24,2 is used three times as a factor.

In the factorization of 36,2 is used twice as a factor.

In the factorization of 120, 2 is used three times as a factor.

It is the greatest power of 2 that is found in all three numbers.

Exercise C In the factorization of 24, 3 is used once as a factor.

In the factorization of 36, 3 is used twice as a factor.

In the factorization of 120, 3 is used once as a factor.

It is the greatest power of the common prime factor 3 that is used in all three numbers.

Page 77 (cont.)
Answers for Exercise D $2^2 \times 3$, or 12

Yes; because it has as a factor each prime factor of 24, 36, 120, and it has as a factor the greatest power of each of these prime factors that is found in all three numbers.

Exercise E $14 = 2 \times 7$. $28 = 2^2 \times 7$.

$21 = 3 \times 7$.

7

Exercise F 7^1 is a common factor of 14, 21, and 28.

7^1, or 7, is the greatest common factor of 14, 21, and 28.

Exercise G $100 = 2^2 \times 5^2$.

$125 = 5^3$.

$75 = 3 \times 5^2$.

Therefore 5^2, or 25, is the greatest common factor of 100, 125, and 75.

Page 78
Answers for "On Your Own" exercises

1 a {1, 2, 3, 6}
b {1, 2, 4, 8}
c {1, 2, 3, 4, 6, 12}
d {1, 3, 5, 15}
e {1, 2, 4, 8, 16}
f {1, 3, 7, 21}
g {1, 2, 3, 5, 6, 10, 15, 30}
h {1, 3, 9, 27}
i {1, 2, 4, 7, 14, 28}
j {1, 2, 3, 4, 6, 9, 12, 18, 36}
k {1, 3, 5, 9, 15, 45}
l {1, 2, 3, 4, 6, 8, 9, 12, 18, 24, 36, 72}

Answers for "On Your Own" exercises

2 a {1, 2}
b {1, 2, 4}
c {1, 3, 5, 15}
d {1, 3}
e {1, 2, 4}
f {1, 3}
g {1, 2, 4}
h {1, 2, 3, 6}
i {1, 3, 5, 15}

3 a 2 f 3
b 4 g 4
c 15 h 6
d 3 i 15
e 4

4 a 5 e 3
b 6 f 4
c 12 g 8
d 9 h 10

5 1 is the answer to each question.

6 Yes; the common factor is 1.

7 {8, 27, 125} This is a typical answer. There are others that would also be correct.

8 9 and 10 are relatively prime because their only common factor is 1.

9 and 18 are not relatively prime, because 9 and 3, as well as 1, are common factors of 9 and 18.

9 1, because a prime number has no factors other than itself and 1.

10 a $24 = 2^3 \times 3$.
$60 = 2^2 \times 3 \times 5$.

Answers for "On Your Own" exercises

10 Greatest common factor: 12

b $36 = 2^2 \times 3^2$.
$90 = 2 \times 3^2 \times 5$.
Greatest common factor: 18

c $72 = 2^3 \times 3^2$.
$108 = 2^2 \times 3^3$.
Greatest common factor: 36

d $25 = 5^2$.
$125 = 5^3$.
Greatest common factor: 25

e $60 = 2^2 \times 3 \times 5$.
$84 = 2^2 \times 3 \times 7$.
Greatest common factor: 12

f $165 = 3 \times 5 \times 11$.
$234 = 2 \times 3^2 \times 13$.
Greatest common factor: 3

g $306 = 2 \times 3^2 \times 17$.
$1173 = 3 \times 17 \times 23$.
Greatest common factor: 51

h $2040 = 2^3 \times 3 \times 5 \times 17$.
$2184 = 2^3 \times 3 \times 7 \times 13$.
Greatest common factor: 24

11 a 15_{seven}
b 5_{seven}
c 11_{seven}

12 a 5
b 3

Answers for "On Your Own" exercises

12 c 3
d 2
e 7
f 11

13 2

14 a 2 d 44
b 14 e 12
c 6 f 9

15 Yes, because the greatest common factor of any two natural numbers is a natural number.

The operation is commutative, because the greatest common factor of $\underline{a}$ and $\underline{b}$ is the same number as the greatest common factor of $\underline{b}$ and $\underline{a}$, for each replacement of $\underline{a}$ and $\underline{b}$ by counting numbers.

$48 * 50 = 2$, and $50 * 48 = 2$ are true statements, if $*$ means that $\underline{a} * \underline{b}$ is the greatest common factor of $\underline{a}$ and $\underline{b}$.

$9 * 27 = 9$, and $27 * 9 = 9$ are true statements, if $*$ means that $\underline{a} * \underline{b}$ is the greatest common factor of $\underline{a}$ and $\underline{b}$.

PART FIVE: APPENDIX[1]

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Glossary of Symbols

		Pages
$\downarrow$ or $\uparrow$	used to show a matching between members of two sets	1
$\updownarrow$	used to show a matching between members of two sets that can be put in one-to-one correspondence	3
{a, b}	"the set whose members are a and b"; used when tabulating or describing sets	3
{ }	the empty set	7
$\underline{a} = \underline{n}\{A\}$	"$\underline{a}$ is the number of members in set A"	7
$<$	is less than	11
$>$	is greater than	11
$\wedge$	the connective "and" in a compound condition	16
$\cup$	union	15

Glossary of Symbols (cont.)

		Pages
$K \cup L$	the union of set K and set L	15
+	plus sign	
a + b	the number replacing b was added to the number replacing a	18
$n(K \cup L)$	the number of members in the union of set K and set L	17
-	minus sign	
a - b	the number replacing b was subtracted from the number replacing a	20
$A \times B$	the Cartesian set $A \times B$; read "A cross B"	23
$n(A \times B)$	the number of members in A cross B	24
(9,8)	the ordered pair whose first component is 9 and whose second component is 8	19
$\times$	multiplication sign	
$a \times b$	the number replacing b was multiplied by the number replacing a	24
$\div$	division sign	
$a \div b$	the number replacing a was divided by the number replacing b	26
ab	multiplication sign; the number replacing b was multiplied by the number replacing a	32
a(b)	multiplication sign; the number replacing b was multiplied by the number replacing a	32
$a \cdot b$	multiplication sign; the number replacing b was multiplied by the number replacing a	33
( )	parentheses; used to indicate grouping, multiplication, or ordered pairs	35
9 + 7	an expression; a numeral that contains one or more symbols of operation	45
a^n	"a to the nth power"; power notation; a used n times as a factor	74

SCOTT, FORESMAN EXPERIMENTAL MATHEMATICS PROJECT

TEST ON BOOKLET 7 (*The System of Natural Numbers*)

NAME_____

In Exercises 1 through 16, write "T" on the answer line for each sentence that expresses a true statement. Write "F" for each sentence that expresses a false statement.

ANSWER KEY

1	$(20,4)$ and $(5,1)$ are mapped onto the same number by division.	1 T
2	If n is replaced by an even number, then $3n$ will be an even number.	2 T
3	If A is a subset of B, then A and B are disjoint sets.	3 F
4	$10 - 2 \times 3 = (10 - 2)3$.	4 F
5	$n\{5\} < n\{4\}$.	5 F
6	24 has just two prime factors.	6 T
7	The set of even numbers is closed under multiplication.	7 T
8	$A \times B$ and $B \times A$ are equivalent sets.	8 T
9	Zero is the successor of one.	9 F
10	For any set A, the union of A and $\{\ \}$ is A.	10 T
11	$n\{0\} = n\{\ \}$.	11 F
12	If A and B are not disjoint and are not empty, then $n(A) + n(B) > n(A \cup B)$.	12 T
13	$U = N$. $\{x \mid x + 2 = 2 + x\} = \{b \mid 2b = b2\}$.	13 T
14	The solution set of $5(2 + x) = 5(2) + 30$ is the same set as the solution set of $3 + (5 + 6) = (5 + x) + 3$. $U = N$.	14 T
15	A number system involves just one operation.	15 F
16	The greatest common factor of two counting numbers must be less than each number.	16 F

Choose your answers for Exercises 17 through 20 from the list below. Write the letter of your selection at the right of the exercise. A letter may be used more than once. $U = N$.

a 0 **c** Any member of N

b 6 **d** Any member of $\{0, 1, 2, 3, 4, 5\}$

17	What number must be the replacement for a if $(6,a)$ and $(a,6)$ are mapped onto the same natural number by subtraction?	17 b
18	What number must be the replacement for a if $(6,a)$ and $(a,6)$ are mapped onto the same natural number by addition?	18 C
19	What number must be the replacement for a if $(6,a)$ and $(a,6)$ are mapped onto the same natural number by multiplication?	19 C

20 What number must be the replacement for a if $(6,a)$ and $(a,6)$ are mapped onto the same natural number by division? 20 b

A list of some of the properties for addition and multiplication of natural numbers is given below. For each statement expressed in Exercises 21 through 32, choose the property that explains why the statement is true. Write the letter of the property on the answer line at the right of the sentence. An answer may be used more than once.

a Distributive property of multiplication over addition
b Associative property of addition
c Commutative property of addition
d Associative property of multiplication
e Commutative property of multiplication
f Closure under addition **g** Closure under multiplication

21 $3 + 20 = 20 + 3$. 21 c
22 $15 - 4 \cdot 3 = 15 - 3 \cdot 4$. 22 e
23 8×10 is a natural number. 23 g
24 $1 + (2 + 3) = (1 + 2) + 3$. 24 b
25 $5(9 + 2) = (9 + 2)5$. 25 e
26 $5(9 + 2) = 5(9) + 5(2)$. 26 a
27 $9 + 26$ is a natural number. 27 f
28 $5 \cdot 4 + 2 = 4 \cdot 5 + 2$. 28 e
29 $13(9) + 13(2) = 13(9 + 2)$. 29 a
30 $2(9 \cdot 3) = (2 \cdot 9)3$. 30 d
31 $(4 - 1)(6 \div 3) = (6 \div 3)(4 - 1)$. 31 e
32 $(2 + 3)(5 + 7) = (2 + 3)5 + (2 + 3)7$. 32 a

Six standard names of numbers are given below. For each expression in Exercises 33 through 40, choose the standard name that names the same number. Write the letter of your selection on the answer line. An answer may be used more than once.

a 60 **b** 27 **c** 20 **d** 2 **e** 16 **f** 84

33 $12 \times 2 + 3$ 33 b
34 $(15 \div 3) \times 4$ 34 c
35 $3 \times 3 \times 3$ 35 b
36 $2^3 + 12$ 36 c
37 $2(13 - 5)$ 37 e
38 $2^2 \times 3 + 3 \times 5$ 38 b
39 $9 \div 9 + 10 \div 10$ 39 d
40 $7 \times 3 + 3 \times 2$ 40 b

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For Exercises 41 through 45, select the correct answer from the statements expressed in the list below. On the answer line write the letter that labels your selection.

a If $S = \{0\}$ and A is any set, then $n(A \times S) = n(A)$.

b For any disjoint sets A, B, and C, $(A \cup B) \cup C$ is the same set as $A \cup (B \cup C)$.

c For any two disjoint sets A and B, $A \cup B = B \cup A$.

d If $T = \{\ \}$ and A is any set, then $A \cup T = A$.

e For any sets A and B, $n(A \times B) = n(B \times A)$.

41 Which statement could you use to show that multiplication of natural numbers is commutative? **41** e

42 Which statement could you use to show that the product of any natural number and 1 is that natural number? **42** a

43 Which statement could you use to show that addition of natural numbers is commutative? **43** c

44 Which statement could you use to show that addition of natural numbers is associative? **44** b

45 Which statement could you use to show that the sum of any natural number and 0 is that natural number? **45** d

In the remaining exercises, a choice of four answers is given for each question. For each exercise, select the correct answer. On the answer line write the letter that labels your selection.

46 Which of the following sentences would you *not* use to show that the number 2 is greater than the number 1?

a $2 = n\{0, 1\}$ **c** $1 = n\{0\}$.

b $\{0\}$ is a proper subset of $\{0,1\}$. **d** $0 = n\{\ \}$. **46** d

47 Set A is a proper subset of set B. Which of the following sentences does *not* express a true statement about A and B?

a B has more members than A.

b There is a many-to-one mapping of set B onto set A.

c There is a one-to-one correspondence between set A and set B.

d The number associated with set A is less than the number associated with set B. **47** c

48 Which of the sets tabulated below is the set of factors of 18?

a $\{18, 36, 54, \cdots\}$ **c** $\{1, 2, 3\}$

b $\{1, 2, 3, 6, 9, 18\}$ **d** $\{2, 9\}$ **48** b

49 If x is a variable for prime numbers, then $2x$ will be a number that has how many factors?

a 4 **b** 3 **c** 2 **d** 5 **49** a

3 0112 115433978

50 Which of the sets named below is the set of common factors of 12 and 18?

a {2×3, 3}
b {1, 2, 3, 4, 6, 12, 9, 18}
c The set of factors of 12
d The set of factors of 6

50 d

51 We will use the following rule to map the members of A and B: Map each member of A onto 2 times that member; and map each member of B onto the quotient of that member and 2. For which sets will we obtain a one-to-one correspondence?

a A = {2, 4}. B = {4, 8, 16}.
b A = {1, 3, 5}. B = {2, 4, 6}.
c A = {10, 20, 30}. B = {5, 10, 15}.
d A = {4, 8, 16}. B = {8, 16, 32}.

51 d

52 Which of the following could you use to show that the addition mapping is a many-to-one mapping?

a In addition, $(6,2)$ is mapped onto $6+2$.
b In addition, $(6,6)$ is mapped onto 12, and $(10,2)$ is mapped onto 12.
c In addition, $(6,6)$ is mapped onto 12, and $(5,5)$ is mapped onto 10.
d In addition, $(0,0)$ is mapped onto 0, and $(0,10)$ is mapped onto 10.

52 b

53 Which of the sets named below is the set of factors of a prime number?

a {1, 21}
b {1, 2, 3}
c {1, 5}
d {3}

53 c

54 $R = \{0, 1, 2, 3\}$. R is the set of possible remainders when a natural number is divided by what number?

a 1
b 2
c 3
d 4

54 d

55 What is the greatest power of 2 that is a common factor of 24, 100, and 148?

a 2^1
b 2^2
c 2^3
d 2^4

55 b

56 Which of the following could you use to show that the operation of addition does *not* distribute over the operation of multiplication in the set of natural numbers?

a $3(5) + 6 = 15 + 18$.
b $3(5 + 6) = 8 \times 9$.
c $3 + (5 \times 6) = (3 + 5)(3 + 6)$.
d $3 \times 6 + 5 \times 6 = 9 \times 30$.

56 c